Multivariate Interpolation: Preserving and Exploiting Symmetry

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Abstract

Interpolation is a prime tool in algebraic computation while symmetry is a qualitative feature that can be more relevant to a mathematical model than the numerical accuracy of the parameters. The article shows how to exactly preserve symmetry in multivariate interpolation while exploiting it to alleviate the computational cost. We revisit minimal degree and least interpolation with symmetry adapted bases, rather than monomial bases. For a space of linear forms invariant under a group action, we construct bases of invariant interpolation spaces in blocks, capturing the inherent redundancy in the computations. With the so constructed symmetry adapted interpolation bases, the uniquely defined interpolant automatically preserves any equivariance the interpolation problem might have. Even with no equivariance, the computational cost to obtain the interpolant is alleviated thanks to the smaller size of the matrices to be inverted.

Keywords: Multivariate Interpolation; Vandermonde matrix; Collocation matrix; Symmetry; Representation Theory.

1. Introduction

Preserving and exploiting symmetry in algebraic computations is a challenge that has been addressed within a few topics and, mostly, for specific groups of symmetry (Collowald and Hubert, 2015; Faugère and Rahmany, 2009; Faugere and Svartz, 2013; Gatermann, 2000; Gatermann and Parrilo, 2004; Hubert, 2019; Hubert and Labahn, 2012, 2013, 2016; Krick et al., 2017; Riener et al., 2013; Riener and Safey El Din, 2018; Verschelde and Gatermann, 1995). The present article addresses multivariate interpolation in the presence of symmetry.

Symmetry is understood as invariance, or equivariance, under the linear action of a finite group. Frequently arising such group actions are, on the base space, central symmetry, rotations of finite order, permutations on the coordinates, or more generally generated by a set of reflections through hyperplanes. By choosing an appropriate set of coordinates, one can always think of such a group action as a subgroup of the orthogonal group.

Due to its relevance in approximation theory and geometrical modeling, interpolation is a prime topic in algebraic computation. Among the several problems in multivariate interpolation (Gasca and Sauer, 2000; Lorentz, 2000), we focus on the construction of a polynomial interpolation space for a given set of linear forms. This particular problem is addressed for instance in

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(Birkhoff, 1979; De Boor and Ron, 1990, 1992a,b; Fassino and Möller, 2016; Möller and Buchberger, 1982; Möller and Sauer, 2000; Sauer, 1997, 2001). Assuming the space generated by the linear forms is invariant under a group action, we show how to, not only, preserve exactly the symmetry, but also, exploit it throughout the computations.

Lagrange interpolation is what comes to mind when we speak of interpolation. For a set of r points $\xi_1, \ldots, \xi_r$ in n-space, called *nodes*, and r values $\eta_1, \ldots, \eta_r$ Lagrange interpolation consists in finding a n-variate polynomial function p such that $p(\xi_i) = \eta_i$, for $1 \leq i \leq r$. The evaluations at the nodes ξ_i are basic examples of linear forms. The space they generate is invariant under a group action as soon as the nodes form a union of orbits of this group action. The above interpolation problem is invariant if $\eta_i = \eta_j$ whenever ξ_i and ξ_j belong to the same orbit. It is then natural to expect an invariant polynomial as interpolant. Yet, contrary to the univariate case, there is no unique interpolant of minimal degree and the symmetry of the interpolation problem may very well be violated in the computed solution (compare Figure 1b and 1a).

In this article we consider a general set of linear forms; Instead of, or in addition to, fixing the values at the nodes, we could also impose the values of some derivatives, moment functionals, or coefficients in a given basis of functions. The new angle on the above problems that is offered in this article is to consider general set of linear forms invariant under a group action and seek to compute interpolants that respect the *symmetry* of the interpolation problem. We mentioned invariance as an instance of symmetry, but equivariance is the more general concept. An interpolation space for a set of linear forms is a subspace of the polynomial ring that has a unique interpolant for each related interpolation problem, *i.e.,* instantiation of the linear forms. We show that the unique interpolants automatically inherit the symmetry of the problem when the interpolation space is invariant (Section 3). We need to point out that, when much of the litterature on algebraic computation restricts to monomial bases, an invariant interpolation space is generally not spanned by monomials.

A canonical interpolation space, the *least interpolation space*, was introduced by De Boor and Ron (1990, 1992a,b). We shall observe that it is invariant as soon as the space of linear forms is. In floating point arithmetics though, the computed interpolation space might fail to be exactly invariant. Yet, in mathematical modeling, symmetry is often more relevant than numerical accuracy. We shall remedy this flaw and further exploit symmetry to mitigate the cost and numerical sensitivity of computing a minimal degree or least interpolation space.

Minimal degree interpolation spaces can be constructed by Gaussian elimination in a multivariate Vandermonde (or *collocation*) matrix. A different collection of the terms allows one to determine the least interpolation space. The columns of the Vandermonde matrix are traditionally indexed by monomials. We show how any other graded basis of the polynomial ring can be used. When the space of linear forms is invariant under a group, there is then a two fold gain in using a *symmetry adapted basis*. On one hand, the computed interpolation space will be exactly invariant independently of the accuracy of the data for the interpolation problem. On the other hand, the new Vandermonde matrix is block diagonal so that Gaussian elimination can be performed independently on smaller size matrices, with better conditioning. Further computational savings result from identical blocks being repeated according to the dimension of the related irreducible representations of the group. Symmetry adapted bases also played a prominent role in (Collowald and Hubert, 2015; Gatermann and Parrilo, 2004; Riener et al., 2013) where it allowed the block diagonalisation of a multivariate Hankel matrix.

In Section 2 we define minimal degree and least interpolation space and review how to compute a basis of it with Gaussian elimination. In Section 3 we make explicit how, in an interpolation problem, symmetry is expressed and can be preserved. In Section 4 we review symmetry adapted bases and show how the Vandermonde matrix becomes block diagonal in these. This is applied to provide an algorithm for the computation of invariant interpolation spaces in Section 5 together with a selection of relevant invariant and equivariant interpolation problems.

A preliminary version of the material in this paper appeared in the conference proceedings of ISSAC 2019 (Rodriguez Bazan and Hubert, 2019). The results there were restricted to groups with only absolutely irreducible representations. By introducing *real symmetry adapted bases* we can now deal with a group with both absolutely irreducible representations and irreducible representations of complex types. These include the cyclic groups C_m, for $m > 2$. New examples were added accordingly. We also took a more elegant approach to the block diagonalization of the Vandermonde matrix. By introducing the underlying linear map, the block diagonalization is now proved by exhibiting the equivariance of the latter.

An implementation of the algorithms presented in this paper are available in the Maple library SyCo (Symmetery & Computations) developed by the first author. The Maple library can be found at `http://www-sop.inria.fr/members/Evelyne.Hubert/SyCo`, together with a worksheet of all the examples of this paper.

2. Polynomial interpolation

In this section we first review the definitions and constructions of interpolation spaces of minimal degree. After introducing dual polynomial bases we generalize the construction of least interpolation spaces. We shall then be in a position to work with adapted bases to preserve and exploit symmetry.

2.1. Interpolation space

Hereafter, $\mathbb{K}$ denotes either $\mathbb{C}$ or $\mathbb{R}$. $\mathbb{K}[x] = \mathbb{K}[x_1, \ldots, x_n]$ denotes the ring of polynomials in the variables $x_1, \ldots, x_n$ with coefficients in $\mathbb{K}$; $\mathbb{K}[x]_{\leq d}$ and $\mathbb{K}[x]_d$ the $\mathbb{K}-$vector spaces of polynomials of degree at most d and the space of homogeneous polynomials of degree d respectively.

The *dual* of $\mathbb{K}[x]$, the set of $\mathbb{K}-$linear forms on $\mathbb{K}[x]$, is denoted by $\mathbb{K}[x]^*$. A typical example of a linear form on $\mathbb{K}[x]$ is the evaluation $\mathbb{e}_\xi$ at a point ξ of $\mathbb{K}^n$. It is defined by

$$\begin{array}{rccc} \mathbb{e}_\xi : & \mathbb{K}[x] & \to & \mathbb{K} \\ & p & \mapsto & p(\xi). \end{array}$$

Other examples of linear forms on $\mathbb{K}[x]$ are given by compositions of evaluation and differentiation

$$\begin{array}{rccl} \lambda : & \mathbb{K}[x] & \to & \mathbb{K} \\ & p & \mapsto & \sum_{j=1}^{r} \mathbb{e}_{\xi_j} \circ q_j(\partial)(p), \end{array}$$

with $\xi_j \in \mathbb{K}^n, q_j \in \mathbb{K}[x]$ and $\partial^\alpha = \dfrac{\partial}{\partial x_1^{\alpha_1}} \cdots \dfrac{\partial}{\partial x_n^{\alpha_n}}$.

An *interpolation problem* is a pair (Λ, ϕ) where Λ is a finite dimensional linear subspace of $\mathbb{K}[x]^*$ and $\phi : \Lambda \longrightarrow \mathbb{K}$ is a $\mathbb{K}$-linear map. An *interpolant*, *i.e.,* a solution to the interpolation problem, is a polynomial p such that

$$\lambda(p) = \phi(\lambda) \text{ for any } \lambda \in \Lambda. \tag{2.1}$$

To illustrate the meaning of the above, let us look at how *Lagrange interpolation* can be phrased in these terms. Lagrange interpolation starts with a set of nodes $\xi_1, \ldots, \xi_r$ in $\mathbb{K}^n$ and a set of values $\eta_1, \ldots \eta_r \in \mathbb{K}$, and consists in finding a polynomial p such that $\mathbb{e}_{\xi_j}(p) = \eta_j$, $1 \leq j \leq r$. Then, on one hand, Λ is the linear subspace $\left\langle \mathbb{e}_{\xi_1}, \ldots, \mathbb{e}_{\xi_r} \right\rangle$ spanned by the evaluations at the nodes. On the other hand, $\phi : \Lambda \longrightarrow \mathbb{K}$ is the linear map that takes $\mathbb{e}_{\xi_i}$ to η_i.

An *interpolation space for* Λ is a subspace P of $\mathbb{K}[\mathrm{x}]$ such that Equation (2.1) has a unique solution in P for any map $\phi : \Lambda \longrightarrow \mathbb{K}$.

2.2. Vandermonde matrix

For a given linear space of linear forms Λ we introduce the *Vandermonde operator* w as

$$\begin{array}{rccc} w: & \mathbb{K}[\mathrm{x}] & \rightarrow & \Lambda^* \\ & p & \rightarrow & (\cdot\,,\,p)\,, \end{array} \tag{2.2}$$

where $(\cdot\,,\,\cdot)$ is the dual pairing between Λ and Λ^*, *i.e.*, $(\lambda\,,\,p) = \lambda(p)$. The map w is surjective when Λ is finite dimensional. Indeed, for every $\phi \in \Lambda^*$ let $p_\phi \in \mathbb{K}[\mathrm{x}]$ be a solution of the interpolation problem (Λ, ϕ). Then $\phi = (\cdot, p_\phi)$ and therefore $w(p_\phi) = \phi$.

We denote by $w_d : \mathbb{K}[\mathrm{x}]_{\leq d} \rightarrow \Lambda^*$ the restriction of w to $\mathbb{K}[\mathrm{x}]_{\leq d}$. The matrix of w_d in the bases $\mathcal{P} = \{p_1, p_2, \ldots, p_m\}$ of $\mathbb{K}[\mathrm{x}]_{\leq d}$ and the dual of the basis $\mathcal{L} = \{\lambda_1, \lambda_2, \ldots, \lambda_r\}$ of Λ is the *Vandermonde matrix*

$$\mathrm{W}_{\mathcal{L}}^{\mathcal{P}} := \left[\lambda_i\left(p_j\right)\right]_{\substack{1\leq i\leq r\\ 1\leq j\leq m}}. \tag{2.3}$$

As in the univariate case, the Vandermonde matrix appears naturally in multivariate interpolation. Indeed $\langle p_1, \ldots, p_r\rangle$ is an interpolation space for $\langle \lambda_1, \ldots, \lambda_r\rangle$ if and only if for any $\phi : \Lambda \rightarrow \mathbb{K}$ there exists a unique $a = (a_1, \ldots, a_r)^T \in \mathbb{K}^r$ such that $p = a_1 p_1 + \ldots a_r p_r$ is a solution of (Λ, ϕ). Then a is a solution of the linear system $\mathrm{W}_{\mathcal{L}}^{\mathcal{P}}\; a = (\phi(\lambda_1), \ldots, \phi(\lambda_r))^T$. Therefore $\langle p_1, \ldots, p_r\rangle$ is an interpolation space if and only if $\mathrm{W}_{\mathcal{L}}^{\mathcal{P}}$ is an invertible matrix.

The above observation leads to a straightforward approach to compute an interpolation space for $\langle \lambda_1, \ldots, \lambda_r\rangle$. Since the elements of $\mathcal{L}$ are linearly independent, there is $d > 0$ such that $\mathrm{W}_{\mathcal{L}}^{\mathcal{P}_d}$ has full row rank, where $\mathcal{P}_d$ is a basis of $\mathbb{K}[\mathrm{x}]_{\leq d}$. For Lagrange interpolation $d \leq |\mathcal{L}|$. Hence we can choose r linearly independent columns $j_1, j_2, \ldots j_r$ of $\mathrm{W}_{\mathcal{L}}^{\mathcal{P}_d}$ and the corresponding space $P = \left\langle p_{j_1}, \ldots p_{j_r} \right\rangle$ is an interpolation space for Λ.

In order to select r linearly independent columns of $\mathrm{W}_{\mathcal{L}}^{\mathcal{P}_d}$ we can use any rank revealing decomposition of $\mathrm{W}_{\mathcal{L}}^{\mathcal{P}_d}$. Singular value decomposition (SVD) and *QR* decomposition provide better numerical accuracy but to obtain a minimal degree interpolation space we shall resort to Gaussian elimination. It produces a LU factorization of $\mathrm{W}_{\mathcal{L}}^{\mathcal{P}_d}$ where L is an invertible matrix and $\mathrm{U} = \left[u_{ij}\right]_{\substack{1\leq i\leq r\\ 1\leq j\leq m}}$ is in row echelon form. This means that there exists an increasing sequence $j_1, \ldots, j_r$ with $j_i \geq i$, such that u_{ij_i} is the first non-zero entry in the i−th row of U. We call $j_1, \ldots, j_r$ the *echelon index sequence* of $\mathrm{W}_{\mathcal{L}}^{\mathcal{P}_d}$. They index a maximal set of linearly independent columns of $\mathrm{W}_{\mathcal{L}}^{\mathcal{P}_d}$.

2.3. Minimal degree

It is desirable to build an interpolation space such that the degree of the interpolating polynomials be as small as possible. We shall use the definition of minimal degree solution for an interpolation problem defined by De Boor and Ron (1992a,b); Sauer (1998).

Definition 2.1. *An interpolation space P for Λ is of minimal degree if for any other interpolation space Q for Λ*

$$\dim(Q \cap \mathbb{K}[\mathrm{x}]_{\leq d}) \leq \dim(P \cap \mathbb{K}[\mathrm{x}]_{\leq d}), \forall d \in \mathbb{N}.$$

When you have an interpolation space P for Λ of minimal degree, the interpolant obtained as the unique solution in P of an interpolation problem (Λ, ϕ) will always be of the lowest possible degree (Sauer, 1998).

We say that a countable set of homogeneous polynomials $P = \{p_1, p_2, \ldots\}$ is *ordered by degree* if $i \leq j$ implies that $\deg p_i \leq \deg p_j$.

Proposition 2.2. *Let $\mathcal{L}$ be a basis of Λ. Let $\mathcal{P}_d$, $d > 0$, be a homogeneous basis of $\mathbb{K}[\mathrm{x}]_{\leq d}$ ordered by degree, such that $\mathrm{W}_{\mathcal{L}}^{\mathcal{P}_d}$ has full row rank. Let $j_1, \ldots, j_r$ be the echelon sequence of $\mathrm{W}_{\mathcal{L}}^{\mathcal{P}_d}$ obtained by Gaussian elimination with partial pivoting. Then $P := \left\langle p_{j_1}, \ldots, p_{j_r} \right\rangle$ is a minimal degree interpolation space for Λ.*

Proof. Let Q be another interpolation space for Λ. Let $q_1, q_2 \ldots q_m$ be a basis of $Q \cap \mathbb{K}[\mathrm{x}]_{\leq e}$ with $e \leq d$. Since $\mathcal{P}_d$ is a homogeneous basis of $\mathbb{K}[\mathrm{x}]_{\leq d}$, any q_i can be written as a linear combination of elements of $\mathcal{P}_d \cap \mathbb{K}[\mathrm{x}]_{\leq e}$. Considering $q_i = \sum_j a_{ji} p_j$ we get that $\lambda(q_i) = \sum_j a_{ji} \lambda(p_j)$ for any $\lambda \in \Lambda$.

Let $\{p_{j_{i_1}}, p_{j_{i_2}}, \ldots p_{j_{i_n}}\}$ be the elements of P that form a basis of $P \cap \mathbb{K}[\mathrm{x}]_{\leq e}$. Gaussian elimination on $\mathrm{W}_{\mathcal{L}}^{\mathcal{P}_d}$ ensures that $\lambda(b)$ is a linear combination of $\lambda(p_{j_{i_1}}), \ldots \lambda(p_{j_{i_n}})$ for any $b \in \mathcal{P}_d \cap \mathbb{K}[\mathrm{x}]_{\leq d}$ and $\lambda \in \Lambda$. The latter implies that $\lambda(q_i) = \sum_{k=1}^{n} c_{ki} \lambda(p_{j_{i_k}})$ for $1 \leq i \leq m$ and $c_{ki} \in \mathbb{K}$. If $m > n$ then the matrix $\mathrm{C} = \left(c_{ij}\right)_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}}$ has linearly independent columns, and therefore there exist $d_1, d_2, \ldots d_m \in \mathbb{K}$ such that $\sum_{i=1}^{m} d_i \lambda(q_i) = \lambda(\sum_{i=1}^{m} d_i q_i) = 0$ for any $\lambda \in \Lambda$ which is a contradiction with the fact that Q is an interpolation space of Λ. Then we can conclude that $m \leq n$ and P is a minimal degree interpolation space for Λ. □

2.4. Duality and apolar product

$\mathbb{K}[\mathrm{x}]^*$ can be identified with the ring of formal power series $\mathbb{K}[[\partial]]$ through the isomorphism $\Phi : \mathbb{K}[[\partial]] \longrightarrow \mathbb{K}[\mathrm{x}]^*$, where for $p = \sum_\alpha p_\alpha x^\alpha \in \mathbb{K}[\mathrm{x}]$ and $f = \sum_{\alpha \in \mathbb{N}^n} f_\alpha \partial^\alpha \in \mathbb{K}[[\partial]]$

$$\Phi(f)(p) := \sum_{\alpha \in \mathbb{N}^n} f_\alpha \frac{\partial^\alpha p}{\partial x^\alpha}(0) = \sum_{\alpha \in \mathbb{N}} \alpha! f_\alpha p_\alpha, \quad \text{where, for } \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n,\ \alpha! = \alpha_1! \ldots \alpha_n!.$$

For instance, the evaluation $\mathbb{e}_\xi$ at a point $\xi \in \mathbb{K}^n$ is represented by $\mathrm{e}^{(\xi, \partial)} = \sum_{k \in \mathbb{N}} \frac{(\xi, \partial)^k}{k!}$, the power series expansion of the exponential function with frequency ξ. The dual pairing

$$\begin{array}{ccc} \mathbb{K}[\mathrm{x}]^* \times \mathbb{K}[\mathrm{x}] & \to & \mathbb{K} \\ (\lambda, p) & \to & \lambda(p) \end{array}$$

induces the *apolar product* on $\mathbb{K}[\mathrm{x}]$ by associating $p \in \mathbb{K}[\mathrm{x}]$ to $\overline{p}(\partial) \in \mathbb{K}[[\partial]]$. For $p = \sum_\alpha p_\alpha x^\alpha$ and $q = \sum_\alpha q_\alpha x^\alpha$ the apolar product between p and q is given by $\langle p, q \rangle := \overline{p}(\partial) q = \sum_\alpha \alpha! \overline{p}_\alpha q_\alpha \in \mathbb{K}$. Note that for a linear map $a : \mathbb{K}^n \to \mathbb{K}^n$, $\langle p, q \circ a \rangle = \left\langle p \circ \overline{a}^t, q \right\rangle$.

For a homogeneous basis $\mathcal{P} = \{p_1, p_2, \ldots\}$ of $\mathbb{K}[\mathrm{x}]$, or subspace thereof, we define the dual basis $\mathcal{P}^\dagger$ to be the set of homogeneous polynomials $\{p_1^\dagger, p_2^\dagger, \ldots\}$ such that $\left\langle p_i^\dagger, p_j \right\rangle = \delta_{ij}$. For instance the dual basis of the monomial basis $\{x^\alpha\}_{\alpha \in \mathbb{N}^n}$ is $\{\frac{1}{\alpha!} x^\alpha\}_{\alpha \in \mathbb{N}^n}$. Thus any linear form $\lambda \in \mathbb{K}[\mathrm{x}]^*$ can be written as $\lambda = \sum_{\alpha \in \mathbb{N}^n} \frac{1}{\alpha!} \lambda(x^\alpha) \partial^\alpha \in \mathbb{K}[[\partial]]$. More generally, any linear form on $\langle \mathcal{P} \rangle$ can be written as $\lambda = \sum_{p \in \mathcal{P}} \lambda(p) \overline{p}^\dagger(\partial)$.

2.5. Least interpolation space

For a space of linear forms $\Lambda \subset \mathbb{K}[\mathrm{x}]^*$, a canonical interpolation space $\Lambda_\downarrow$ was introduced by De Boor and Ron (1992b). It has a desirable set of properties. An algorithm to build a basis of $\Lambda_\downarrow$ based on Gaussian elimination on the Vandermonde matrix is presented in (De Boor and Ron, 1992a). In this algorithm the authors consider the Vandermonde matrix associated to the monomial basis of $\mathbb{K}[\mathrm{x}]$. The notion of dual bases introduced above allows us to extend the algorithm to any graded basis of $\mathbb{K}[\mathrm{x}]$.

The initial form of a power series $\lambda \in \mathbb{K}[[\partial]]$, denoted by $\lambda_\downarrow \in \mathbb{K}[\mathrm{x}]$ in (De Boor and Ron, 1990, 1992a,b), is the unique homogeneous polynomial for which $\lambda - \lambda_\downarrow(\partial)$ vanishes to highest possible order at the origin. Given a linear space of linear forms Λ, we define $\Lambda_\downarrow$ as the linear span of all $\lambda_\downarrow$ with $\lambda \in \Lambda$.

Proposition 2.3. *Let $\mathcal{P} = \{p_1, p_2, \ldots\}$ be a homogeneous basis of $\mathbb{K}[\mathrm{x}]$ ordered by degree and $\mathcal{L} = \{\lambda_1, \ldots, \lambda_r\}$ be a basis of Λ. Let $\mathrm{LU} = \mathrm{W}_{\mathcal{L}}^{\mathcal{P}}$ be the factorization of $\mathrm{W}_{\mathcal{L}}^{\mathcal{P}}$ provided by Gaussian elimination with partial pivoting and $\{j_1, j_2, \ldots, j_r\}$ its echelon index sequence. If $\mathrm{U} = (u_{ij})$ consider, for $1 \le i \le r$,*

$$q_i = \sum_{\deg(p_k)=\deg(p_{j_i})} u_{ik}\, \overline{p}_k^\dagger \tag{2.4}$$

where $\mathcal{P}^\dagger = \{p_1^\dagger, \ldots, p_j^\dagger, \ldots\}$ is the dual basis of $\mathcal{P}$ with respect to the apolar product. Then $Q = \{q_1, \ldots q_r\}$ is a basis for $\Lambda_\downarrow$.

Proof. Let $\mathrm{L}^{-1} = (a_{ij})$ and consider $\varsigma_l = \sum_{j\in\mathbb{N}} u_{lj}\overline{p}_j^\dagger(\partial)$. Since $u_{lj} = \sum_{i=1}^{r} a_{li}\lambda_i(p_j)$, then

$$\varsigma_\ell = \sum_{j\in\mathbb{N}} \Big(\sum_{i=1}^{r} a_{li}\lambda_i(p_j) \Big) \overline{p}_j^\dagger(\partial) = \sum_{i=1}^{r} a_{li} \sum_{j\in\mathbb{N}} \lambda_i(p_j)\overline{p}_j^\dagger(\partial) = \sum_{i=1}^{r} a_{li}\lambda_i \in \Lambda.$$

Notice that $q_l = \varsigma_{l\downarrow}$ and therefore $q_l \in \Lambda_\downarrow$ for $1 \le l \le r$.

The j_i are strictly increasing so that $\{q_1, q_2, \ldots, q_r\} \subset \Lambda_\downarrow$ are linearly independent. By (De Boor and Ron, 1992b, Proposition 2.10) we have that $r = \dim \Lambda = \dim \Lambda_\downarrow$. Thus we conclude that Q is a basis of $\Lambda_\downarrow$. □

In (Rodriguez Bazan and Hubert, 2020, 2021) we provide an alternative construction of the least interpolation space. Proceeding degree by degree, with a QR decomposition at each step, we compute there an orthogonal basis of the least interpolation space. As argued in (Fassino and Möller, 2016), where is determined an interpolation space of minimal degree, the use of QR decompositions instead of a LU factorisations entails a better numerical stability.

3. Symmetry

We define the concepts of *invariant interpolation problem* (IIP) and *equivariant interpolation problem* (EIP). These interpolation problems have a structure that we want to be preserved by the interpolant. We show that this is automatically achieved when choosing the interpolant in an invariant interpolation space. Then the solution of an IIP is an invariant polynomial and the solution of an EIP is an equivariant polynomial map. In Section 5 we show that the least interpolation space is invariant and how to better compute an invariant interpolation space of minimal degree.

The symmetries we shall deal with are given by the linear action of a finite group $\mathcal{G}$ on $\mathbb{K}^n$. It is thus given by a representation $\varrho : \mathcal{G} \to \mathrm{GL}_n(\mathbb{K})$ of $\mathcal{G}$ on $\mathbb{K}^n$. It induces a representation ρ of $\mathcal{G}$ on $\mathbb{K}[\mathrm{x}]$ given by:

$$\text{For } p \in \mathbb{K}[\mathrm{x}],\ \rho(g)\, p = p \circ \varrho(g^{-1}), \quad \textit{i.e.,}\ (\rho(g)\, p)(x) = p\left(\varrho(g^{-1})x\right). \tag{3.1}$$

It also induces a linear representation on the space of linear forms, the dual representation of ρ :

$$(\rho^*(g)\lambda)\,(p) = \lambda(\rho(g^{-1})p) = \lambda(p \circ \varrho(g)), \quad g \in \mathcal{G},\ p \in \mathbb{K}[\mathrm{x}] \text{ and } \lambda \in \mathbb{K}[\mathrm{x}]^*. \tag{3.2}$$

We shall deal with an invariant subspace Λ of $\mathbb{K}[\mathrm{x}]^*$. Hence the restriction of ρ^* to Λ is a linear representation of $\mathcal{G}$ in Λ.

3.1. Invariance

A Lagrange interpolation problem is $\mathcal{G}$-invariant if the nodes in $\mathbb{K}^n$ form a union of orbits for the representation $\varrho : \mathcal{G} \to \mathrm{GL}_n(\mathbb{K})$ and the values are equal for nodes on the same orbit. In other words, given $\xi_1, \ldots, \xi_m$ with orbits $O_1, \ldots, O_m$ and $\eta_1, \ldots, \eta_m \in \mathbb{K}^n$, an interpolant $p \in \mathbb{K}[\mathrm{x}]$ is to satisfy $p\,(\varrho(g)\xi_k) = \eta_k$ for any $g \in \mathcal{G}$. This is generalized as follow.

Definition 3.1. *Let Λ be a space of linear forms and $\phi : \Lambda \longrightarrow \mathbb{K}$ a linear map. The pair (Λ, ϕ) defines an* invariant interpolation problem *if*

1. *Λ is closed under the action of $\mathcal{G}$.*
2. $\phi(\rho^*(g)(\lambda)) = \phi(\lambda)$ for any $g \in \mathcal{G}$ and $\lambda \in \Lambda$.

It is natural to expect that the solution to an invariant interpolation problem is an invariant polynomial. Yet, not all minimal degree interpolants are invariant.

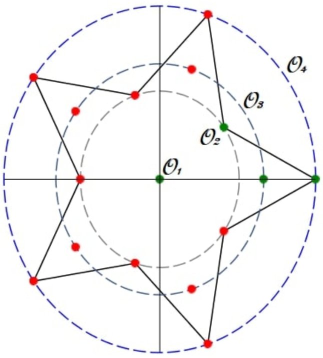
Example 3.1.1. The dihedral group D_m is a group of order $2m$ whose elements can be written $\{s^\epsilon r^k \mid \epsilon \in \{0,1\}, 0 \le k < m\}$, the generators s and r being of respective order 2 and m. We consider its representation ϱ in $\mathbb{R}^2$ defined by the reflection $\varrho(s)$ and the rotation $\varrho(r)$ given by:

$$\varrho(s) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \varrho(r) = \begin{pmatrix} \cos\left(\frac{2\pi}{m}\right) & -\sin\left(\frac{2\pi}{m}\right) \\ \sin\left(\frac{2\pi}{m}\right) & \cos\left(\frac{2\pi}{m}\right) \end{pmatrix} \tag{3.3}$$

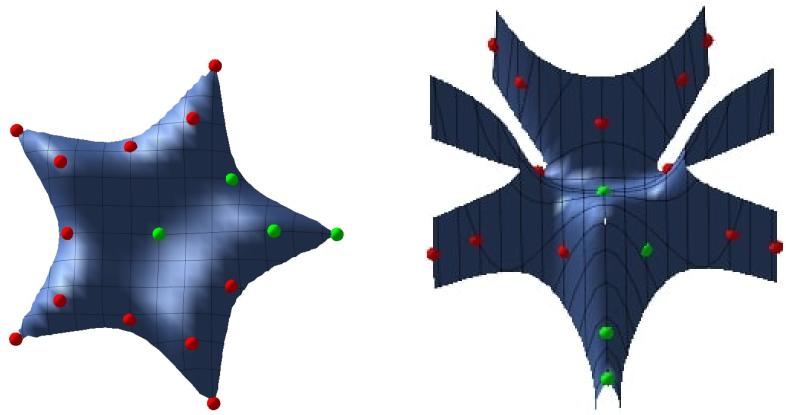
This representation is the symmetry group of the planar regular m-gon. We now take $m = 5$.

Consider $\Xi \subset \mathbb{R}^2$ a set of $1 + 3 \times 5$ points illustrated on Figure 1. They form four orbits O_1, O_2, O_3, O_4 of D_5 so that $\Lambda := \operatorname{span}(\mathbb{e}_\xi \mid \xi \in \Xi)$ is invariant. An invariant interpolation problem is given by the pair (Λ, ϕ) where ϕ is defined by $\phi(\mathbb{e}_\xi) = \frac{1}{10}$ if $\xi \in O_1$, $\phi(\mathbb{e}_\xi) = 0$ if $\xi \in O_2 \cup O_4$, and $\phi(\mathbb{e}_\xi) = -\frac{1}{2}$ if $\xi \in O_3$. We show in Figure 1a the graph of the interpolant that could be expected, and in Figure 1b the graph of another interpolant of minimal degree, obtained from a monomial basis; The D_5 symmetry is not respected in this latter.

Proposition 3.2. *Let (Λ, ϕ) be an invariant interpolation problem. If P is an invariant interpolation space and let $p \in \mathbb{K}[\mathrm{x}]$ is the solution of (Λ, ϕ) in P, then p is an invariant polynomial.*



	#Nodes	A node per orbit
O_1	1	$\xi_1 = (0, 0)$
O_2	5	$\xi_2 = (0.1934557, 0.1405538)$
O_3	5	$\xi_7 = (0.4695268, 0)$
O_4	5	$\xi_{12} = (0.6260358, 0)$



(a) invariant interpolant of minimal degree (b) interpolant obtained from a monomial basis

Figure 1: Invariant Lagrange interpolation problem

Proof. For any $\lambda \in \Lambda$ and $g \in \mathcal{G}$ we have that $\lambda(p) = \phi(\lambda)$ and $\rho^*(g)(\lambda)p = \phi(\rho^*(g)(\lambda))$. Since ϕ is $\mathcal{G}-$invariant, we get that

$$\lambda(\rho(g^{-1})p) = \rho^*(g)(\lambda)p = \phi(\rho^*(g)(\lambda)) = \phi(\lambda) = \lambda(p)$$

for any $\lambda \in \Lambda$. The latter implies that $\rho(g^{-1})p - p \in \cap_{\lambda\in\Lambda} \ker \lambda$. As P is $\mathcal{G}$-invariant, $\rho(g^{-1})p - p \in \cap_{\lambda\in\Lambda} \ker \lambda \bigcap P$. As P is an interpolation space for Λ we have that $\cap_{\lambda\in\Lambda} \ker \lambda \bigcap P = \{0\}$ and therefore we can conclude that $\rho(g^{-1})p = p$ for any $g \in \mathcal{G}$, *i.e.,* p is invariant. □

3.2. *Equivariance*

Equivariant maps define, for instance, dynamical systems that exhibit particularly interesting patterns and are relevant to model physical or biological phenomena (Chossat and Lauterbach, 2000; Golubitsky et al., 1988). In this context, it is interesting to have a tool to offer equivariant maps that interpolate some observed local behaviors.

Let $\mathbb{K}[\mathrm{x}]^m$ be the module of polynomial mappings with m components, and let $\theta : \mathcal{G} \longrightarrow \mathrm{GL}_m(\mathbb{K})$ be a linear representation on $\mathbb{K}^m$. A polynomial mapping $f = (f_1, f_2, \ldots, f_m)^t$ is $\varrho - \theta$ *equivariant* if $f(\varrho(g)x) = \theta(g)f(x)$ for any $g \in \mathcal{G}$. The space of equivariant mappings over $\mathbb{K}$, denoted by $\mathbb{K}[\mathrm{x}]^\theta_\varrho$, is a $\mathbb{K}[\mathrm{x}]^\mathcal{G}-$module, where $\mathbb{K}[\mathrm{x}]^\mathcal{G}$ is the ring of invariant polynomials.

Definition 3.3. *Let Λ be a space of linear forms on $\mathbb{K}[\mathrm{x}]$ and $\phi : \Lambda \longrightarrow \mathbb{K}^m$ a linear map. The pair (Λ, ϕ) defines a $\varrho - \theta$* equivariant interpolation problem *if*

1. *Λ is closed under the action of $\mathcal{G}$.*
2. $\phi(\rho^*(g)(\lambda)) = \theta(g)\phi(\lambda)$ for any $g \in \mathcal{G}$ and $\lambda \in \Lambda$.

The solution of an EIP (Λ, ϕ), is a polynomial map $f = (f_1, \ldots, f_m)^t$ such that $\lambda(f) = (\lambda(f_1), \ldots, \lambda(f_m))^t = \phi(\lambda)$ for any $\lambda \in \Lambda$. It is natural to seek f as an equivariant map. It is remarkable that any type of equivariance will be respected as soon as the interpolation space is invariant.

Proposition 3.4. *Let (Λ, ϕ) be an $\varrho - \theta$ equivariant interpolation problem. Let P be an invariant interpolation space for Λ and let $f = (f_1, \ldots, f_m)^t$ be the solution of (Λ, ϕ) in P. Then $f \in \mathbb{K}[x]^\theta_\varrho$, i.e., f is a $\varrho - \theta$ equivariant polynomial mapping.*

Proof. We need to prove that $f \circ \varrho(g^{-1}) = \theta(g)f$. As $f \in P^m$ and P is invariant, $f \circ \varrho(g^{-1})$ belongs to P^m as does $\theta(g)f$. As P is an interpolation space for Λ, it is thus enough to prove that $\lambda\,(f \circ \varrho(g)) = \lambda\,(\theta(g)f)$ for all $\lambda \in \Lambda$.

On one hand
$$\lambda\,(f \circ \varrho(g)) = (\rho^*(g)\lambda)\,(f) = \phi(\rho^*(g)\lambda),$$
where the first equality is by definition of ρ^* and the second one stems from f being a solution of the interpolation problem. On the other hand
$$\lambda\,(\theta(g)f) = \theta(g)\lambda\,(f) = \theta(g)\phi(\lambda) = \phi\,(\rho^*(g)\lambda)\,,$$
where the first equality is by linearity, the second one stems from f being a solution of the interpolation problem, and the third one comes from the definition of an equivariant interpolation problem. □

Example 3.2.1. The cyclic group C_m is a subgroup of D_m order m that is generated by a single element r. Its representation $\tau : C_m \to \mathrm{GL}_2(\mathbb{R})$ as a group of rotations is given by
$$\tau(r) = \begin{pmatrix} \cos\left(\frac{2\pi}{m}\right) & -\sin\left(\frac{2\pi}{m}\right) \\ \sin\left(\frac{2\pi}{m}\right) & \cos\left(\frac{2\pi}{m}\right) \end{pmatrix}. \tag{3.4}$$

We consider equivariant interpolation problems for the groups D_3 and C_3 acting on $\mathbb{R}^2$. The representations of D_3 and C_3 in $\mathbb{R}^2$ are given in Equations (3.3) and (3.4) respectively.

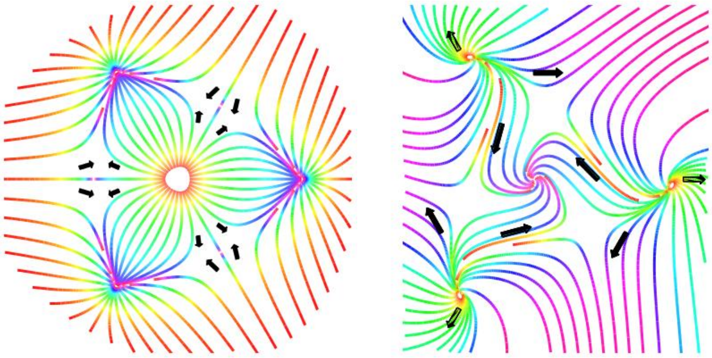
For D_3 we consider the space Λ_D of linear forms spanned by the evaluations at the set of nodes consisting of the two orbits of the points $\xi_1 = (-\frac{5\sqrt{3}}{3}, \frac{1}{3})^t$ and $\xi_2 = (-\sqrt{3}, \frac{1}{3})^t$. We define $\phi_D : \Lambda \to \mathbb{R}^2$ by $\phi_D(\mathrm{e}_{\varrho(g)\xi_1}) = \varrho(g)\begin{pmatrix} v_1 \\ v_3 \end{pmatrix}$ and $\phi_D(\mathrm{e}_{\varrho(g)\xi_2}) = \varrho(g)\begin{pmatrix} v_2 \\ v_4 \end{pmatrix}$.

For C_3 we consider Λ_C spanned by the evaluations at the points of the orbits of $\zeta_1 = \left(-\frac{3}{2}, 0\right)^t$, $\zeta_2 = \left(0, \frac{5}{2}\right)^t$ and $\zeta_3 = \left(\frac{7}{2}, 0\right)^t$. We define $\phi_C : \Lambda \to \mathbb{R}^2$ by
$$\phi_C(\mathrm{e}_{\tau(g)\zeta_1}) = \tau(g)\begin{pmatrix} u_1 \\ u_4 \end{pmatrix}, \quad \phi_C(\mathrm{e}_{\tau(g)\zeta_2}) = \tau(g)\begin{pmatrix} u_2 \\ u_5 \end{pmatrix} \quad \text{and} \quad \phi_C(\mathrm{e}_{\tau(g)\zeta_3}) = \tau(g)\begin{pmatrix} u_3 \\ u_6 \end{pmatrix}.$$

The thus defined interpolation problems are clearly equivariant. For each quadruplet $v \in \mathbb{R}^4$ and sextuplet $u \in \mathbb{R}^6$ it is desirable to find interpolants $(p_1, p_2)^t \in \mathbb{R}[x]^2$ and $(q_1, q_2)^t \in \mathbb{R}[x]^2$ that are $\varrho - \varrho$ and $\tau - \tau$ equivariant mappings respectively. This will define the equivariant dynamical systems

$$\left\{ \begin{array}{l} \dot{x_1}(t) = p_1(x_1(t), x_2(t)), \\ \dot{x_2}(t) = p_2(x_1(t), x_2(t)); \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{l} \dot{x_1}(t) = q_1(x_1(t), x_2(t)), \\ \dot{x_2}(t) = q_2(x_1(t), x_2(t)). \end{array} \right.$$

The sets of integral curves, limit cycles and equilibrium points for these systems will exhibit the D_3 and C_3 symmetries respectively. In Figures 2a and 2b we draw the integral curves of the equivariant vector fields thus constructed. The data of the interpolation problem are illustrated by the black arrows



(a) D_3 (b) C_3

Figure 2: Integral curves for the equivariant vector fields interpolating the equivariant set of vectors shown in black.

4. Symmetry reduction

In this section we show how, when the space Λ of linear forms is invariant, the Vandermonde matrix can be made block diagonal, in *symmetry adapted bases*. This block diagonalisation of the Vandermonde matrix indicates how computation can be organized more efficiently, and robustly. It just draws on the invariance of the space of linear forms. So, when the evaluation points can be chosen, it makes sense to introduce symmetry among them.

The block diagonalization stems from an equivariance of the Vandermonde operator that we exhibit in this section. Beforehand we recall though the construction of symmetry adapted bases. They are first constructed over $\mathbb{C}$, as in (Serre, 1977; Fässler and Stiefel, 1992), and then combined to obtain *real symmetry adapted bases*.

4.1. Symmetry adapted bases

We are in a setting where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. A *linear representation* of the group $\mathcal{G}$ on the $\mathbb{K}-$vector space V is a group morphism $\mathfrak{r} : \mathcal{G} \to \mathrm{GL}(V)$, the group of isomorphisms from V to itself. V is called the *representation space*. If V has finite dimension n, upon introducing a basis $\mathcal{P}$ of V the isomorphism $\mathfrak{r}(g)$ can be described by a non-singular $n \times n$ matrix over $\mathbb{K}$. This *representing matrix* is denoted by $[\mathfrak{r}(g)]_{\mathcal{P}}$. The function $\chi : \mathcal{G} \longrightarrow \mathbb{K}$, with $\chi(g) \to \mathrm{trace}(\mathfrak{r}(g))$ is the *character of the representation* $\mathfrak{r}$.

The *dual* or *contragredient representation* of $\mathfrak{r}$ is the representation $\mathfrak{r}^*$ on the dual vector space V^* defined by:

$$\mathfrak{r}^*(g)(\lambda) = \lambda \circ \mathfrak{r}(g^{-1}) \text{ for any } \lambda \in V^*. \tag{4.1}$$

If $\mathcal{P}$ is a basis of V and $\mathcal{P}^*$ its dual basis then $[\mathfrak{r}^*(g)]_{\mathcal{P}^*} = [\mathfrak{r}(g^{-1})]^t_{\mathcal{P}}$. It follows that $\chi_{\mathfrak{r}^*}(g) = \chi_{\mathfrak{r}}(g^{-1})$.

An inner product is $\mathcal{G}-$invariant with respect to a linear representation $\mathfrak{r}$ if

$$\langle v, w\rangle = \langle \mathfrak{r}(g)(v), \mathfrak{r}(g)(w)\rangle \text{ for any } g \in \mathcal{G} \text{ and } v, w \in V.$$

If V is finite dimensional, we can always find an invariant product. In a basis that is orthonormal w.r.t. this inner product, the representing matrices of $\mathfrak{r}$ are unitary, if $\mathbb{K} = \mathbb{C}$, or orthogonal if $\mathbb{K} = \mathbb{R}$.

A linear representation $\mathfrak{r}$ of a group $\mathcal{G}$, on a $\mathbb{K}$-vector space V, is *irreducible* if there is no proper nonzero subspace W of V with the property that, for every $g \in \mathcal{G}$, the isomorphism $\mathfrak{r}(g)$ maps every vector of W into W. In this case, its representation space V is also called *irreducible*. The contragredient representation $\mathfrak{r}^*$ is irreducible when $\mathfrak{r}$ is. Any representation of a finite group is completely reducible, meaning that it decomposes into a finite number of irreducible representations.

Morally a symmetry adapted basis $\mathcal{P}$ of V is characterized by the fact that

$$[\mathfrak{r}(g)]_{\mathcal{P}} = \operatorname{diag}\left(\mathfrak{r}^{(1)}(g) \otimes \mathrm{I}_{m_1},\ \ldots,\ \mathfrak{r}^{(\mathfrak{n})}(g) \otimes \mathrm{I}_{m_{\mathfrak{n}}}\right)$$

where $\mathfrak{r}^{(1)}, \ldots, \mathfrak{r}^{(\mathfrak{n})}$ are the inequivalent irreducible representations of $\mathcal{G}$ over $\mathbb{K}$, and $m_1, \ldots, m_{\mathfrak{n}}$ are the multiplicities of these representations in $\mathfrak{r}$. The complete reduction of the representation $\mathfrak{r}$ and its representation space are denoted by $\mathfrak{r} = m_1\mathfrak{r}^{(1)} \oplus \cdots \oplus m_{\mathfrak{n}}\mathfrak{r}^{(\mathfrak{n})}$ and $V = V^{(1)} \oplus \cdots \oplus V^{(\mathfrak{n})}$. Each *isotypic components* $V^{(\ell)}$ is an invariant subspace that is the direct sum of m_ℓ irreducible subspaces and the restriction of $\mathfrak{r}$ to each one is equivalent to $\mathfrak{r}^{(\ell)}$.

We now specialize to $\mathbb{K} = \mathbb{C}$. Let $\mathfrak{r}^{(\ell)}$, $\ell = 1, \ldots, \underline{\mathfrak{n}}$, be the irreducible n_ℓ-dimensional representations of $\mathcal{G}$ over $\mathbb{C}$. With $\chi^{(\ell)}$ the character of $\mathfrak{r}^{(\ell)}$ we determine the multiplicity m_ℓ and the projection $\pi^{(\ell)}$ onto the isotypic component $V^{(\ell)}$

$$m_\ell = \frac{1}{|\mathcal{G}|}\sum_{g\in\mathcal{G}} \overline{\chi^{(\ell)}(g)}\chi(g), \quad \pi^{(\ell)} = \frac{n_\ell}{|\mathcal{G}|}\sum_{g\in\mathcal{G}} \chi^{(\ell)}(g^{-1})\,\mathfrak{r}(g). \tag{4.2}$$

To go further in the decomposition, consider that the irreducible representations are given as matrix representations, *i.e.,* $\mathfrak{r}^{(\ell)} : \mathcal{G} \to \mathrm{GL}_{n_\ell}(\mathbb{C})$ with $\mathfrak{r}^{(\ell)}(g) = \left(r^{(\ell)}_{ij}(g)\right)_{1\le i,j\le n_\ell}$. For $1 \le i, j \le n_\ell$, let

$$\pi^{(\ell)}_{ij} = \frac{n_\ell}{|\mathcal{G}|}\sum_{g\in\mathcal{G}} r^{(\ell)}_{ji}(g^{-1})\,\mathfrak{r}(g). \tag{4.3}$$

Let $\{p^{(\ell)}_1, \ldots, p^{(\ell)}_{m_\ell}\}$ be a basis of the subspace $V^{(\ell,1)} = \pi^{(\ell)}_{11}(V)$. A *symmetry adapted basis* $\mathcal{P}^{(\ell)}$ *of the isotypic component* $V^{(\ell)}$ can be computed constructively as follows:

$$\mathcal{P}^{(\ell)} = \{p^{(\ell)}_1, \ldots, p^{(\ell)}_{m_\ell}, \ldots, \pi^{(\ell)}_{n_\ell 1}(p^{(\ell)}_1), \ldots, \pi^{(\ell)}_{n_\ell 1}(p^{(\ell)}_{m_\ell})\}. \tag{4.4}$$

The union $\mathcal{P}$ of the bases $\mathcal{P}^{(\ell)}$ of $V^{(\ell)}$, is a *symmetry adapted basis* for V. Indeed, by (Serre, 1977, Proposition 8), the set $\{\pi^{(\ell)}_{k1}(p^{(\ell)}_1), \ldots, \pi^{(\ell)}_{k1}(p^{(\ell)}_{m_\ell})\}$ is a basis of $V^{(\ell,k)} = \pi^{(\ell)}_{kk}(V)$ and $V^{(\ell)} = V^{(\ell,1)} \oplus \cdots \oplus V^{(\ell,n_\ell)}$. Furthermore, for $1 \le i \le m_\ell$, $\left\{p^{(\ell)}_i, \pi^{(\ell)}_{21}(p^{(\ell)}_i), \ldots, \pi^{(\ell)}_{n_\ell 1}(p^{(\ell)}_i)\right\}$ is a basis of an irreducible subspace with representation $\mathfrak{r}^{(\ell)}$.

As we can perceive in (4.4), a symmetry adapted basis of a vector space V is not unique. Yet it is fully determined by the choice of the bases for the subspaces $V^{(\ell,1)}$ with $1 \leq \ell \leq \underline{\mathfrak{n}}$. Hereafter we denote by $\mathcal{P}^{(\ell,k)}$ the polynomial map defined by

$$\mathcal{P}^{(\ell,k)} = \left(\pi^{(\ell)}_{k1}(p^{(\ell)}_1), \dots, \pi^{(\ell)}_{k1}(p^{(\ell)}_{m_\ell})\right). \tag{4.5}$$

Proposition 4.1. *If $\mathcal{P} = \cup_{\ell=1}^{\mathfrak{n}} \mathcal{P}^{(\ell)}$ is a symmetry adapted basis of V where $\mathcal{P}^{(\ell)}$ spans the isotypic component associated to $\mathfrak{r}^{(\ell)}$, then its dual basis $\mathcal{P}^* = \cup_{i=1}^{\mathfrak{n}} (\mathcal{P}^*)^{(\ell)}$ in V^* is a symmetry adapted basis where $(\mathcal{P}^*)^{(\ell)}$ spans the isotypic component associated to $\left(\mathfrak{r}^{(\ell)}\right)^*$.*

Indeed if $[\mathfrak{r}(g)]_{\mathcal{P}} = \operatorname{diag}\left(\mathfrak{r}^{(1)}(g) \otimes \mathrm{I}_{m_1},\ \dots,\ \mathfrak{r}^{(\mathfrak{n})}(g) \otimes \mathrm{I}_{m_\mathfrak{n}}\right)$ then $[\mathfrak{r}^*(g)]_{\mathcal{P}^*} = \operatorname{diag}\left(\left(\mathfrak{r}^{(\ell)}(g)\right)^{-t} \otimes \mathrm{I}_{m_\ell} \mid \ell = 1..\mathfrak{n}\right)$.

Corollary 4.2. *If $\mathcal{P}$ is a symmetry adapted basis of $\mathbb{K}[\mathrm{x}]_{\leq d}$, so is its dual $\mathcal{P}^\dagger$ with respect to the apolar product.*

Theorem 4.3. *Let ϑ and θ be representations of $\mathcal{G}$ on the vector space V and W respectively, and $\mathcal{P}$ and $\mathcal{Q}$ respective symmetry adapted bases of V and W. Consider $\phi : V \longrightarrow W$ an $\vartheta - \theta$ equivariant map,* i.e., *$\phi \circ \vartheta(g) = \theta(q) \circ \phi$ for all $g \in \mathcal{G}$. The matrix Φ of ϕ in the bases $\mathcal{P}$ and $\mathcal{Q}$ has the following structure*

$$\Phi = \operatorname{diag}\left(\mathrm{I}_{n_\ell} \otimes \Phi^{(\ell)} \mid i = 1 \dots \mathfrak{n}\right), \tag{4.6}$$

where $\Phi^{(\ell)}$ is a $b_\ell \times a_\ell$ matrix, a_ℓ and b_ℓ being the multiplicities of $\mathfrak{r}^{(\ell)}$ in V and W respectively.

This is a consequence of Schur's lemma (Serre, 1977, Proposition 4). The proof is a simple extension of (Fässler and Stiefel, 1992, Theorem 2.5) for equivariant endomorphisms.

If we consider unitary representing matrices $\mathfrak{r}^{(\ell)}(g)$, and an orthonormal basis $\{p^{(\ell)}_1, \dots, p^{(\ell)}_{m_\ell}\}$ of $V^{(\ell,1)}$ with respect to a $\mathcal{G}-$invariant inner product, then the above process leads to an *orthonormal symmetry adapted basis* (Fässler and Stiefel, 1992, Theorem 5.4).

4.2. Real symmetry adapted bases

Some groups have irreducible representations over $\mathbb{C}$ that have no representing matrices in $\mathbb{R}$. This is the case for the cyclic group C_m, $m > 2$. This would lead to symmetry adapted bases whose components are polynomials in $\mathbb{C}[x]$. This is not desirable for an interpolation problem over $\mathbb{R}$. Fortunately one can determine a *real symmetry adapted basis* by combining the isotypic components related to conjugate irreducible representations.

Any linear representation $\mathfrak{r}$ on a $\mathbb{R}-$vector space V can be considered as a linear representation on the $\mathbb{C}-$vector space $V \otimes_{\mathbb{R}} \mathbb{C}$, i.e., the vector space obtained from V by extending the scalars from the real numbers to the complex numbers. Consider a representation $\mathfrak{r}$ that is irreducible over $\mathbb{R}$. This representation can be of three types (Serre, 1977, Chapter 13.2):

Absolutely irreducible The representation $\mathfrak{r}$ is still irreducible when considered over $\mathbb{C}$.

Complex type When considered over $\mathbb{C}$ the representation $\mathfrak{r}$ splits into two conjugate, non equivalent, irreducible representations over $\mathbb{C}$. Their characters are thus not real.

Quaternonian type When considered over $\mathbb{C}$ the representation $\mathfrak{r}$ splits in two equivalent irreducible representations over $\mathbb{C}$, whose character is real.

We shall make the reasonable assumption that the group considered has no irreducible representation of quaternionian type. We shall denote $\mathfrak{a}$ the number of absolutely irreducible representations of the group $\mathcal{G}$ and $\mathfrak{c}$ the number of real irreducible representations of complex type. Hence the number of irreducible real representations is $\mathfrak{n} = \mathfrak{a} + \mathfrak{c}$ while the number of irreducible complex representations $\underline{\mathfrak{n}} = \mathfrak{a} + 2\mathfrak{c}$. When needed the (complex) irreducible representations will be accordingly denoted $\mathfrak{r}^{(1)}, \ldots, \mathfrak{r}^{(\mathfrak{a})}$, and $\mathfrak{r}^{(\mathfrak{a}+1)}, \ldots, \mathfrak{r}^{(\mathfrak{a}+\mathfrak{c})}$ together with their conjugates $\bar{\mathfrak{r}}^{(\mathfrak{a}+1)}, \ldots, \bar{\mathfrak{r}}^{(\mathfrak{a}+\mathfrak{c})}$.

The construction of a real symmetry adapted basis for a represenation $\mathfrak{r}$ on a $\mathbb{R}$-vector space V is based on the construction of a complex symmetry adapted basis as presented in Section 4.1. From a basis of the isotypic component associated to $\mathfrak{r}^{(\ell)}$, for $1 \leq \ell \leq \mathfrak{a} + \mathfrak{c}$ we construct a basis for the associated real irreducible representation.

Absolutely irreducible. In this case we can choose representing matrices $\left(r_{ij}^{(\ell)}(g)\right)_{1 \leq i,j \leq n}$ of the irreducible representation $\mathfrak{r}^{(\ell)}$ with real entries. Taking a real basis $\{p_1^{(\ell)}, \ldots, p_{m_\ell}^{(\ell)}\}$ of the subspace $\pi_{11}^{(\ell)}(V \otimes_{\mathbb{R}} \mathbb{C})$, the basis

$$\mathcal{P}^{(\ell)} = \{p_1^{(\ell)}, \ldots, p_{m_\ell}^{(\ell)}, \pi_{21}^{(\ell)}(p_1^{(\ell)}), \ldots, \pi_{21}^{(\ell)}(p_{m_\ell}^{(\ell)}), \ldots, \pi_{n_\ell 1}^{(\ell)}(p_1^{(\ell)}), \ldots, \pi_{n_\ell 1}^{(\ell)}(p_{m_\ell}^{(\ell)})\} \tag{4.7}$$

is a real symmetry adapted basis of $V^{(\ell)}$.

Complex Type. Consider the pair of complex conjugate irreducible represenations $\mathfrak{r}^{(\ell)}$ and $\bar{\mathfrak{r}}^{(\ell)}$ together with their characters $\chi^{(\ell)}$ and $\bar{\chi}^{(\ell)}$. They have the same multiplicity m_ℓ in $\mathfrak{r}$. If

$$\widehat{\mathcal{P}}^{(\ell)} = \{p_1^{(\ell)}, \ldots, p_{m_\ell}^{(\ell)}, \pi_{21}^{(\ell)}(p_1^{(\ell)}), \ldots, \pi_{21}^{(\ell)}(p_{m_\ell}^{(\ell)}), \ldots, \pi_{n_\ell 1}^{(\ell)}(p_1^{(\ell)}), \ldots, \pi_{n_\ell 1}^{(\ell)}(p_{m_\ell}^{(\ell)})\}$$

is a symmetry adapted basis of $V^{(\ell)}$, a real symmetry adapted basis $\mathcal{P}^{(\ell)}$ of the space $V^{(\ell)} \oplus \overline{V^{(\ell)}}$ is given by

$$\begin{aligned}\mathcal{P}^{(\ell)} = \Big\{ & p_1^{(\ell)} + \overline{p}_1^{(\ell)},\ \tfrac{1}{\mathrm{i}}\left(p_1^{(\ell)} - \overline{p}_1^{(\ell)}\right),\ \ldots,\ p_{m_\ell}^{(\ell)} + \overline{p}_{m_\ell}^{(\ell)},\ \tfrac{1}{\mathrm{i}}\left(p_{m_\ell}^{(\ell)} - \overline{p}_{m_\ell}^{(\ell)}\right);\ \ldots\ldots; \\ & \pi_{n_\ell 1}^{(\ell)}\left(p_1^{(\ell)}\right) + \overline{\pi}_{n_\ell 1}^{(\ell)}\left(\overline{p}_1^{(\ell)}\right), \tfrac{1}{\mathrm{i}}\left(\pi_{n_\ell 1}^{(\ell)}\left(p_1^{(\ell)}\right) - \overline{\pi}_{n_\ell 1}^{(\ell)}\left(\overline{p}_1^{(\ell)}\right)\right), \ldots, \pi_{n_\ell 1}^{(\ell)}\left(p_{m_\ell}^{(\ell)}\right) + \overline{\pi}_{n_\ell 1}^{(\ell)}\left(\overline{p}_{m_\ell}^{(\ell)}\right), \tfrac{1}{\mathrm{i}}\left(\pi_{n_\ell 1}^{(\ell)}\left(p_{m_\ell}^{(\ell)}\right) - \overline{\pi}_{n_\ell 1}^{(\ell)}\left(\overline{p}_{m_\ell}^{(\ell)}\right)\right)\Big\},\end{aligned} \tag{4.8}$$

Considering

$$b_k^{(\ell)} = \begin{cases} p_k^{(\ell)} + \bar{p}_k^{(\ell)} & \text{if } k = 1 \mod 2 \\ \frac{1}{\mathrm{i}}\left(p_k^{(\ell)} - \bar{p}_k^{(\ell)}\right) & \text{otherwise} \end{cases}$$

and $\hat{\pi}_{ij}^{(\ell)} = \pi_{ij}^{(\ell)} + \overline{\pi}_{ij}^{(\ell)}$ we can write (4.8) similarly to (4.7) as follows

$$\mathcal{P}^{(\ell)} = \left\{b_1^{(\ell)}, \ldots, b_{2m_\ell}^{(\ell)};\ \hat{\pi}_{21}^{(\ell)}\left(b_1^{(\ell)}\right), \ldots, \hat{\pi}_{21}^{(\ell)}\left(b_{2m_\ell}^{(\ell)}\right);\ \ldots;\ \hat{\pi}_{n_\ell 1}^{(\ell)}\left(b_1^{(\ell)}\right), \ldots, \hat{\pi}_{n_\ell 1}^{(\ell)}\left(b_{2m_\ell}^{(\ell)}\right)\right\}. \tag{4.9}$$

If $\widehat{\mathcal{P}}^{(\ell)}$ is an orthonormal symmetry adapted basis of $V^{(\ell)}$ then:

$$\langle \pi_{i1}^{(\ell)}(b_k) \pm \overline{\pi}_{i1}^{(\ell)}(\overline{b_k}),\ \pi_{i1}^{(\ell)}(b_l) \pm \overline{\pi}_{i1}^{(\ell)}(\overline{b_l}) \rangle = \mp 2\delta_{kl}. \tag{4.10}$$

Hence $\frac{1}{\sqrt{2}}\mathcal{P}^{(\ell)}$ is an orthonormal real symmetry adapted basis of $V^{(\ell)} \oplus \bar{V}^{(\ell)}$.

When the group $\mathcal{G}$ admits absolutely irreducible representations and irreducible representations of complex type, a real symmetry adapted basis for a representation space V is characterized by the fact that

$$[\mathfrak{r}(g)]_{\mathcal{P}} = \operatorname{diag}\left(\mathrm{A}^{(1)}(g),\ \ldots,\ \mathrm{A}^{(\mathfrak{a}+\mathfrak{c})}(g)\right),$$

and $\mathrm{A}^{(\ell)}(g)$ is given by

$$\mathrm{A}^{(\ell)}(g) = \begin{cases} \mathfrak{r}^{(\ell)}(g) \otimes \mathrm{I}_{m_\ell} & \text{if} \quad 1 \le \ell \le \mathfrak{a} \\ \begin{pmatrix} \mathrm{I}_{m_\ell} \otimes \mathrm{B}^{(\ell)}_{11}(g) & \mathrm{I}_{m_\ell} \otimes \mathrm{B}^{(\ell)}_{12}(g) & \cdots & \mathrm{I}_{m_\ell} \otimes \mathrm{B}^{(\ell)}_{1n_\ell}(g) \\ \mathrm{I}_{m_\ell} \otimes \mathrm{B}^{(\ell)}_{21}(g) & \mathrm{I}_{m_\ell} \otimes \mathrm{B}^{(\ell)}_{21}(g) & \cdots & \mathrm{I}_{m_\ell} \otimes \mathrm{B}^{(\ell)}_{2n_\ell}(g) \\ \cdots & \cdots & & \cdots \\ \mathrm{I}_{m_\ell} \otimes \mathrm{B}^{(\ell)}_{n_\ell 1}(g) & \mathrm{I}_{m_\ell} \otimes \mathrm{B}^{(\ell)}_{n_\ell 2}(g) & \cdots & \mathrm{I}_{m_\ell} \otimes \mathrm{B}^{(\ell)}_{n_\ell n_\ell}(g) \end{pmatrix} & \text{if} \quad \mathfrak{a} < \ell \le \mathfrak{a} + \mathfrak{c} \end{cases}$$

with $\mathrm{B}^{(\ell)}_{ij}(g) = \begin{pmatrix} s^{(\ell)}_{ij}(g) & -t^{(\ell)}_{ij}(g) \\ t^{(\ell)}_{ij}(g) & s^{(\ell)}_{ij}(g) \end{pmatrix}$, where

$$\mathrm{S}^{(\ell)}(g) = \left(s^{(\ell)}_{ij}\right)_{ij} = \frac{1}{2}\left(\mathfrak{r}^{(\ell)}(g) + \bar{\mathfrak{r}}^{(\ell)}(g)\right) \quad \text{and} \quad \mathrm{T}^{(\ell)}(g) = \left(t^{(\ell)}_{ij}\right)_{ij} = \frac{\mathrm{i}}{2}\left(\mathfrak{r}^{(\ell)}(g) - \bar{\mathfrak{r}}^{(\ell)}(g)\right).$$

Note that, had we ordered the basis elements as $b^{(\ell)}_k = p^{(\ell)}_k + \bar{p}^{(\ell)}_k$ and $b^{(\ell)}_{m_\ell+k} = \frac{1}{\mathrm{i}}\left(p^{(\ell)}_k - \bar{p}^{(\ell)}_k\right)$, for $1 \le k \le m_\ell$, the representation matrix of $\mathfrak{r}(g)$ would more simply have been

$$\begin{bmatrix} \mathrm{S}^{(\ell)}(g) & -\mathrm{T}^{(\ell)}(g) \\ \mathrm{T}^{(\ell)}(g) & \mathrm{S}^{(\ell)}(g) \end{bmatrix} \otimes \mathrm{I}_{m_\ell}.$$

Our choice is motivated by the fact that, when V is a space of polynomials and $\hat{\mathcal{P}}^{(\ell)}$ is ordered by degree, so is $\mathcal{P}^{(\ell)}$.

Proposition 4.4. *Let ϑ and θ be representations of $\mathcal{G}$ on the $\mathbb{R}-$vector space V and W respectively, with real symmetry adapted bases $\mathcal{P}$ and $\mathcal{Q}$. Consider $\psi : V \longrightarrow W$ a $\vartheta - \theta$ equivariant map. Then the matrix Ψ of ψ in the bases $\mathcal{P}$ and $\mathcal{Q}$ has the following structure*

$$\Psi = \operatorname{diag}\left(\mathrm{I}_{n_\ell} \otimes \Psi^{(\ell)} \mid \ell = 1 \ldots \mathfrak{a} + \mathfrak{c}\right). \tag{4.11}$$

If $\mathcal{P}$ and $\mathcal{Q}$ stem from the symmetry adapted bases $\widehat{\mathcal{P}}$ and $\widehat{\mathcal{Q}}$ of $V \otimes_{\mathbb{R}} \mathbb{C}$ and $W \otimes_{\mathbb{R}} \mathbb{C}$ respectively and $\Phi = \operatorname{diag}\left(\mathrm{I}_{n_\ell} \otimes \Phi^{(\ell)} \mid \ell = 1 \ldots \mathfrak{a} + 2\mathfrak{c}\right)$ is the matrix in w.r.t $\widehat{\mathcal{P}}$ and $\widehat{\mathcal{Q}}$ of the $\vartheta - \theta$ equivariant map $\phi : V \otimes_{\mathbb{R}} \mathbb{C} \longrightarrow W \otimes_{\mathbb{R}} \mathbb{C}$, given by $\phi(zv) \to \mathrm{Re}(z)\psi(v) + i\,\mathrm{Im}(z)\psi(v)$. Then for every $1 \le \ell \le \mathfrak{a} + \mathfrak{c}$ the matrix $\Psi^{(\ell)}$ has the following structure

$$\Psi^{(\ell)} = \begin{cases} \Phi^{(\ell)} & \text{if} \quad 1 \le \ell \le \mathfrak{a} \\ \begin{pmatrix} s^{(\ell)}_{11} & -t^{(\ell)}_{11} & \ldots & s^{(\ell)}_{1a_\ell} & -t^{(\ell)}_{1a_\ell} \\ t^{(\ell)}_{11} & s^{(\ell)}_{11} & \ldots & t^{(\ell)}_{1a_\ell} & s^{(\ell)}_{1a_\ell} \\ \cdots & \cdots & & \cdots & \cdots \\ s^{(\ell)}_{b_\ell 1} & -t^{(\ell)}_{b_\ell 1} & \ldots & s^{(\ell)}_{b_\ell a_\ell} & -t^{(\ell)}_{b_\ell a_\ell} \\ t^{(\ell)}_{b_\ell 1} & s^{(\ell)}_{b_\ell 1} & \ldots & t^{(\ell)}_{b_\ell a_\ell} & s^{(\ell)}_{b_\ell a_\ell} \end{pmatrix} & \text{if} \quad \mathfrak{a} < \ell \le \mathfrak{a} + \mathfrak{c} \end{cases},$$

where a_ℓ and b_ℓ are the multiplicities of $\mathfrak{r}^{(\ell)}$ *in* $V \otimes_{\mathbb{R}} \mathbb{C}$ *and* $W \otimes_{\mathbb{R}} \mathbb{C}$ *respectively and the matrices* $\mathrm{S}^{(\ell)} = \left(s_{ij}^{(\ell)}\right)_{\substack{1\le i\le b_\ell \\ 1\le j\le a_\ell}}$ *and* $\mathrm{T}^{(\ell)} = \left(t_{ij}^{(\ell)}\right)_{\substack{1\le i\le b_\ell \\ 1\le j\le a_\ell}}$ *are defined by*

$$\mathrm{S}^{(\ell)} = \frac{1}{2}\left(\Phi^{(\ell)} + \overline{\Phi^{(\ell)}}\right) \qquad \mathrm{T}^{(\ell)} = \frac{\mathrm{i}}{2}\left(\Phi^{(\ell)} - \overline{\Phi^{(\ell)}}\right).$$

Proof. For every real vector v we have that $\psi(v) = \phi(v)$. We distinguish now two cases:

- $v_{ik}^{(\ell)}$ is an element of a symmetry adapted basis associated to an irreducible component of absolutely irreducible type. Then $\psi(v_{ik}^{(\ell)}) = \phi(v_{ik}^{(\ell)}) = \sum_{j=1}^{b_\ell} \Phi_{ji}^{(\ell)} w_{ij}^{(\ell)}$ and the structure of $\Psi^{(\ell)}$ for $1 \le \ell \le \mathfrak{a}$ follows
- $v_{ik}^{(\ell)}$ is an element of a symmetry adapted basis associated to an irreducible component of complex type. Then the elements of the real symmetry adapted basis $\mathcal{P}^{(\ell)}$ for $V^{(\ell)} \oplus \overline{V^{(\ell)}}$ are given by Equation (4.9). The following qualities holds

$$\psi\left(v_{ik}^{(\ell)} + \overline{v}_{ik}^{(\ell)}\right) = \sum_{j=1}^{b_\ell} \Phi_{ji}^{(\ell)} w_{ij}^{(\ell)} + \sum_{j=1}^{b_\ell} \overline{\Phi}_{ji}^{(\ell)} \overline{w}_{ij}^{(\ell)} = \sum_{j=1}^{b_\ell} \left\{ s_{ji}^{(\ell)}\left(w_{ij}^{(\ell)} + \overline{w}_{ij}^{(\ell)}\right) + \frac{1}{\mathrm{i}} t_{ji}^{(\ell)}\left(w_{ij}^{(\ell)} - \overline{w}_{ij}^{(\ell)}\right)\right\}$$

$$\psi\left(\frac{1}{\mathrm{i}}\left(v_{ik}^{(\ell)} - \overline{v}_{ik}^{(\ell)}\right)\right) = \sum_{j=1}^{b_\ell} \left\{ -t_{ji}^{(\ell)}\left(w_{ij}^{(\ell)} + \overline{w}_{ij}^{(\ell)}\right) + \frac{1}{\mathrm{i}} s_{ji}^{(\ell)}\left(w_{ij}^{(\ell)} - \overline{w}_{ij}^{(\ell)}\right)\right\}$$

  and therefore the structure of $\Psi^{(\ell)}$ follows for $\mathfrak{a} \le \ell \le \mathfrak{a} + \mathfrak{c}$.

□

Conventions. We introduce the following conventions so as to have uniform statements for symmetry adapted bases of both real and complex vectors spaces. In the complex case $\mathfrak{n} = \mathfrak{a} + 2\mathfrak{c}$ is the number of inequivalent irreducible representations of the group $\mathcal{G}$ and m_ℓ is the multiplicity of $\mathfrak{r}^{(\ell)}$ in the representation considered. In the real case $\mathfrak{n} = \mathfrak{a} + \mathfrak{c}$ is the number of inequivalent real irreducible representations of the group $\mathcal{G}$ and m_ℓ is, in the representation considered,

- the multiplicity of $\mathfrak{r}^{(\ell)}$ if $1 \le \ell \le \mathfrak{a}$;
- twice the multiplicity of $\mathfrak{r}^{(\ell)}$ if $\mathfrak{a} + 1 \le \ell \le \mathfrak{a} + \mathfrak{c}$.

In either case we denote a symmetry adapted basis by $\mathcal{P} = \cup_{\ell=1}^{\mathfrak{n}} \mathcal{P}^{(\ell)}$ and say that $\mathcal{P}^{(\ell)}$ is determined by $p_1^{(\ell)}, \ldots, p_{m_\ell}^{(\ell)}$ to mean that the basis of $\mathcal{P}^{(\ell)}$ is

- *In the complex case:* $p_1^{(\ell)}, \ldots, p_{m_\ell}^{(\ell)};\ \pi_{21}^{(\ell)}(p_1^{(\ell)}), \ldots, \pi_{21}^{(\ell)}(p_{m_\ell}^{(\ell)});\ \ldots,\ \pi_{n_\ell 1}^{(\ell)}(p_1^{(\ell)}), \ldots, \pi_{n_\ell 1}^{(\ell)}(p_{m_\ell}^{(\ell)});$
- *In the real case and* $1 \le \ell \le \mathfrak{a}$ *:* same as above;
- *In the real case and* $\mathfrak{a}+1 \le \ell \le \mathfrak{a}+\mathfrak{c}$ *:* $p_1^{(\ell)}, \ldots, p_{m_\ell}^{(\ell)};\ \hat{\pi}_{21}^{(\ell)}(p_1^{(\ell)}), \ldots, \hat{\pi}_{21}^{(\ell)}(p_{m_\ell}^{(\ell)});\ \ldots;\ \hat{\pi}_{n_\ell 1}^{(\ell)}(p_1^{(\ell)}), \ldots, \hat{\pi}_{n_\ell 1}^{(\ell)}(p_{m_\ell}^{(\ell)}).$

4.3. Block diagonal Vandermonde matrix

We consider a linear representation ϱ of a finite group $\mathcal{G}$ on $\mathbb{K}^n$. It induces the representations ρ and its dual ρ^* on the space $\mathbb{K}[\mathrm{x}]$ and $\mathbb{K}[\mathrm{x}]^*$, as made explicit in Equations (3.1) and (3.2).

Proposition 4.5. *Consider θ the restriction of ρ^* to the invariant subspace Λ of $\mathbb{K}[\mathrm{x}]^*$, and θ^* the dual representation on Λ^*.*

The Vandermonde operator $w : \mathbb{K}[\mathrm{x}] \to \Lambda^*$ defined in (2.2) is $\rho - \theta^*$ equivariant.

Proof. This is mostly a matter of unrolling the definitions. We want to show that $w(\rho(g)(p)) = \theta^*(g)\,(w(p))$. The left hand side applied to any $\lambda \in \Lambda$ is equal to $\lambda\,(\rho(g)(p)) = \left(\rho^*(g^{-1})(\lambda)\right)(p)$. The right-hand side applied to any $\lambda \in \Lambda$ is equal to $w(p)\left(\theta(g^{-1})(\lambda)\right) = \left(\theta(g^{-1})(\lambda)\right)(p)$. The conclusion follows since $\theta(g^{-1})(\lambda) = \rho^*(g^{-1})(\lambda)$ by definition of θ. □

Corollary 4.6. *Let $\mathcal{P} = \cup_{\ell=1}^{\mathfrak{n}} \mathcal{P}^{(\ell)}$ and $\mathcal{L} = \cup_{\ell=1}^{\mathfrak{n}} \mathcal{L}^{(\ell)}$ be symmetry adapted bases of $\mathbb{K}[\mathrm{x}]_{\leq d}$ and Λ respectively where*

- *$\mathcal{P}^{(\ell)}$ determined by $\{p_1^{(\ell)}, \ldots, p_{m_\ell}^{(\ell)}\}$ spans the isotypic component associated with the irreducible representation $\mathfrak{r}^{(\ell)}$*
- *$\mathcal{L}^{(\ell)}$ determined by $\{\lambda_1^{(\ell)}, \ldots, \lambda_{r_\ell}^{(\ell)}\}$ spans the isotypic component associated with the irreducible representation $\left(\mathfrak{r}^{(\ell)}\right)^*$*

The Vandermonde matrix $\mathrm{W}_{\mathcal{L}}^{\mathcal{P}}$ is given by

$$\operatorname{diag}\left(\mathrm{I}_{n_\ell} \otimes \left(\lambda_i^{(\ell)}(p_j^{(\ell)})\right)_{\substack{1\leq i\leq r_\ell \\ 1\leq j\leq m_\ell}},\ \ell = 1\ldots\mathfrak{n}\right), \tag{4.12}$$

where $\otimes$ denotes the Kronecker product.

Proof. According to Proposition 4.1, the dual basis $\mathcal{L}^*$ of $\mathcal{L}$ is symmetry adapted with its ℓ-th component being associated to $\mathfrak{r}^{(\ell)}$. The matrix of the Vandermonde operator w in $\mathcal{P}$ and $\mathcal{L}$ is $\mathrm{W}_{\mathcal{L}}^{\mathcal{P}}$. Proposition 4.5 ensures that w is equivariant and thus the result follows from Theorem 4.3. □

Note that $\mathfrak{r}^{(\ell)}$ and $\left(\mathfrak{r}^{(\ell)}\right)^*$ are not equivalent only when $\mathfrak{r}^{(\ell)}$ is of complex type. In particular, when we deal with interpolation over the reals, and thus use a real symmetry adapted basis, there is no distinction to be made.

Example 4.3.1. Let $\mathcal{G}$ be the dihedral group D_3 of order 6. A representation of $\mathcal{G}$ on $\mathbb{R}^2$ is given by Equation (3.3) with $m = 3$. D_3 has three irreducible representations, two of dimension 1 and one of dimension 2.

Consider $\Xi = \{\xi_1, \ldots, \xi_6\}$ the orbit of the point $\xi_1 = \left(-\frac{5\sqrt{3}}{3}, \frac{1}{3}\right)^t$ and let $\Lambda = \operatorname{span}(\mathrm{e}_{\xi_i} \circ D_{\vec{\xi_i}})$, with $D_{\vec{\xi}}$ the directional derivative with direction $\vec{\xi}$. Λ is closed under the action of $\mathcal{G}$. Indeed for any $p \in \mathbb{K}[\mathrm{x}]$, $\rho^*(g)(\mathrm{e}_{\xi_i} \circ D_{\vec{\xi_i}})(p) = \mathrm{e}_{\xi_i} \circ D_{\vec{\xi_i}}(p(\varrho(g^{-1}x)) = \mathrm{e}_{\varrho(g^{-1})\xi_i} \circ D_{\varrho(\vec{g^{-1}})\xi_i}(p(x))$. Since $\varrho(g^{-1})\xi_i = \xi_j$ for some $1 \leq j \leq 6$ we have $\rho^*(g)(\mathrm{e}_{\xi_i} \circ D_{\vec{\xi_i}}) = \mathrm{e}_{\xi_j} \circ D_{\vec{\xi_j}}$.

Considering $\mu_i = \mathfrak{e}_{\xi_i} \circ D_{\vec{\xi_i}}$, $1 \le i \le 6$, a symmetry adapted basis of Λ is given by

$$\mathcal{L} := \{\{\lambda_1\}, \{\lambda_2\}, \{\{\lambda_3, \lambda_4\}, \{\lambda_5, \lambda_6\}\}\},$$

where

$$\begin{array}{ll} \lambda_1 = \mu_1 + \mu_2 + \mu_3 + \mu_4 + \mu_5 + \mu_6, & \lambda_2 = \mu_1 - \mu_2 + \mu_3 - \mu_4 + \mu_5 - \mu_6, \\ \lambda_3 = \mu_1 + \mu_2 - \mu_4 - \mu_5, & \lambda_5 = \frac{\sqrt{3}}{2}(\mu_2 - \mu_1 + \mu_4 + 2\mu_3 - 2\varrho_6 - \varrho_5), \\ \lambda_4 = \mu_3 - \mu_4 - \mu_5 + \mu_6, & \lambda_6 = \frac{\sqrt{3}}{2}(2\mu_2 - 2\mu_1 - \mu_4 + \mu_3 - \mu_5 - \mu_6). \end{array}$$

A symmetry adapted basis of $\mathbb{R}[x]_{\le 3}$ is given by

$$\mathcal{P} := \left\{ \begin{array}{c} \left\{1, x_1^2 + x_2^2, x_1^3 - 3x_1x_2^2\right\}, \\ \left\{x_1^2x_2 - \frac{1}{3}x_2^3\right\}, \\ \left\{\left\{x_1, x_1^2 - x_2^2, x_1^3 + x_1x_2^2\right\}, \left\{x_2, -2x_1x_2, x_1^2x_2 + x_2^3\right\}\right\} \end{array} \right\}.$$

The Vandermonde matrix $\mathrm{W}_{\mathcal{L}}^{\mathcal{P}}$ is block diagonal :

$$\mathrm{W}_{\mathcal{L}}^{\mathcal{P}} = \left(\begin{array}{c|c|cc} \mathrm{A}^{(1)} & & & \\ \hline & \frac{448}{9} & & \\ \hline & & \mathrm{A}^{(3)} & \\ & & & \mathrm{A}^{(3)} \end{array} \right), \quad \begin{array}{l} \mathrm{A}^{(1)} = \begin{pmatrix} 0 & \frac{304}{3} & -240\sqrt{3} \end{pmatrix}, \\ \mathrm{A}^{(3)} = \begin{pmatrix} -\frac{16\sqrt{3}}{3} & \frac{128}{3} & -\frac{1216\sqrt{3}}{9} \\ -\frac{2\sqrt{3}}{3} & -\frac{40}{3} & -\frac{152\sqrt{3}}{9} \end{pmatrix}. \end{array}$$

Example 4.3.2. Let $\mathcal{G}$ be the cyclic group C_3 of order 3. A representation of $\mathcal{G}$ on $\mathbb{R}^2$ is given in (3.4). C_3 has 3 irreducible representations of dimension 1, one that can be realized over $\mathbb{R}$ and a pair of conjugate irreducible representations. The real symmetry adapted bases thus have two components. Consider Λ the space spanned by the orbit of the points ζ_1 and ζ_2 given in Example 3.2.1 . Real symmetry adapted bases of Λ and $\mathbb{R}[x]_{\le 3}$ are given by

$$\mathcal{L} := \left\{ \begin{array}{l} \left\{ \frac{\sqrt{3}}{3}\left(\mathfrak{e}_{\zeta_1} + \mathfrak{e}_{\zeta_2} + \mathfrak{e}_{\zeta_3}\right), \frac{\sqrt{3}}{3}\left(\mathfrak{e}_{\zeta_4} + \mathfrak{e}_{\zeta_5} + \mathfrak{e}_{\zeta_6}\right)\right\}, \\ \left\{ \frac{\sqrt{6}}{6}\left(2\mathfrak{e}_{\zeta_1} - \mathfrak{e}_{\zeta_2} - \mathfrak{e}_{\zeta_3}\right), \frac{\sqrt{6}}{6}\left(2\mathfrak{e}_{\zeta_4} - \mathfrak{e}_{\zeta_5} - \mathfrak{e}_{\zeta_6}\right), \frac{\sqrt{2}}{2}\left(\mathfrak{e}_{\zeta_2} - \mathfrak{e}_{\zeta_3}\right), \frac{\sqrt{2}}{2}\left(\mathfrak{e}_{\zeta_5} - \mathfrak{e}_{\zeta_6}\right) \right\} \end{array} \right\},$$

and

$$\mathcal{P} := \left\{ \begin{array}{c} \left\{1, \frac{x^2}{2} + \frac{y^2}{2}\right\}, \\ \left\{x, -y, \frac{x^2}{2} - \frac{y^2}{2}, -xy\right\} \end{array} \right\}.$$

The Vandermonde matrix $\mathrm{W}_{\mathcal{L}}^{\mathcal{P}}$ is block diagonal :

$$\mathrm{W}_{\mathcal{L}}^{\mathcal{P}} = \begin{pmatrix} \mathrm{A}^{(1)} & \\ & \mathrm{A}^{(2)} \end{pmatrix}, \quad \mathrm{A}^{(1)} = \begin{pmatrix} \sqrt{3} & \frac{9\sqrt{3}}{8} \\ \sqrt{3} & \frac{25\sqrt{3}}{8} \end{pmatrix}, \quad \mathrm{A}^{(2)} = \begin{pmatrix} \frac{3\sqrt{6}}{4} & 0 & \frac{9\sqrt{6}}{16} & 0 \\ 0 & -\frac{5\sqrt{6}}{4} & -\frac{25\sqrt{6}}{16} & 0 \\ 0 & -\frac{3\sqrt{6}}{4} & 0 & -\frac{9\sqrt{6}}{16} \\ -\frac{5\sqrt{6}}{4} & 0 & 0 & \frac{25\sqrt{6}}{16} \end{pmatrix}.$$

5. Equivariant interpolation

In this section we shall first show how to build interpolation spaces of minimal degree that are invariant. We shall actually build symmetry adapted bases for these, exploiting the block diagonal structure of the Vandermonde matrix. Doing so we prove that the least interpolation space is invariant. We then present a selection of invariant or equivariant interpolation problems. As proved in Section 3, the invariance or equivariance is preserved by the interpolant when the interpolation space is invariant. The use of symmetry adapted bases constructed ensures that this equivariance is preserved exactly, independently of the numerical accuracy.

5.1. Constructing invariant interpolation spaces

The starting point is a representation ϱ of $\mathcal{G}$ on $\mathbb{K}^n$ that induces representations ρ and ρ^* on $\mathbb{K}[\mathrm{x}]$ and $\mathbb{K}[\mathrm{x}]^*$. Let Λ be an invariant subspace of $\mathbb{K}[\mathrm{x}]^*$. Hereafter $\mathcal{L}$ is a symmetry adapted basis of Λ and $\mathcal{P}$ a symmetry adapted basis of $\mathbb{K}[\mathrm{x}]_{\leq d}$ consisting of homogeneous polynomials. The elements of $\mathcal{P}$ corresponding to the same irreducible component are ordered by degree.

According to Proposition 4.6, $\mathrm{W}_{\mathcal{L}}^{\mathcal{P}} = \mathrm{diag}\left(\mathrm{I}_{n_\ell} \otimes \mathrm{A}^{(\ell)}\right)$. In the factorization $\mathrm{L}^{(\ell)}\mathrm{U}^{(\ell)} := \mathrm{A}^{(\ell)}$ provided by Gaussian elimination, let $j_1, j_2, \ldots, j_{r_\ell}$ be the echelon index sequence of $\mathrm{U}^{(\ell)}$; r_ℓ is the multiplicity of $\left(\mathrm{r}^{(\ell)}\right)^*$ in Λ. An echelon index sequence for $\mathrm{D}^{(\ell)} = \mathrm{I}_{n_\ell} \otimes \mathrm{A}^{(\ell)}$ is given by

$$J^{(\ell)} = \bigcup_{k=0}^{n_\ell - 1} \{j_1 + km_\ell, j_2 + km_\ell, \ldots, j_{r_\ell} + km_\ell\}.$$

An echelon index sequence of $\mathrm{W}_{\mathcal{L}}^{\mathcal{P}}$ is given by $J = \bigcup_{\ell=1}^{\mathfrak{n}} J^{(\ell)}$. Let $\mathcal{M}^{(\ell)}$ be the set of elements of $\mathcal{P}^{(\ell)}$ that are indexed by elements of $J^{(\ell)}$. From (4.4) we get that

$$\mathcal{M}^{(\ell)} = \{b_{j_1}^{(\ell)}, \ldots, b_{j_{r_\ell}}^{(\ell)};\ \ldots;\ \pi_{n_\ell 1}^{(\ell)}(b_{j_1}^{(\ell)}), \ldots, \pi_{n_\ell 1}^{(\ell)}(b_{j_{r_\ell}}^{(\ell)})\}.$$

We prove the assertions made on the outputs of the algorithm.

Proposition 5.1. *The set of polynomials $\mathcal{M}$ built it in Algorithm 1 spans a minimal degree interpolation space for Λ that is invariant under the action of ρ. $\mathcal{M}$ is furthermore a symmetry adapted basis for this space.*

Proof. Since the elements of $\mathcal{M}$ are the elements of $\mathcal{P}$ indexed by the elements of $J = \bigcup_{\ell=1}^{\mathfrak{n}} J^{(\ell)}$, $\mathrm{W}_{\mathcal{L}}^{\mathcal{M}}$ is invertible. Hence $\mathcal{M}$ spans an interpolation space for Λ. The elements of $\mathcal{M}$ that correspond to the same blocks of $\mathrm{W}_{\mathcal{L}}^{\mathcal{P}}$ are ordered by degree. Thus, as a direct consequence of Proposition 2.2, $\mathcal{M}$ spans a minimal degree interpolation space. We prove now that for any p in $\mathcal{M}$, $\rho(g)(p)$ is in the span of $\mathcal{M}$. By construction we can write $p = \pi_{i1}^{(\ell)}(b)$ for some $1 \leq \ell \leq \mathfrak{n}$, $b \in \mathcal{M}^{(\ell,1)}$, $1 \leq i \leq n_\ell$. By (Serre, 1977, Proposition 8.3) we have that $\rho(g)(p) = \sum_{j=1}^{n_\ell} \mathrm{r}_{ji}^{(\ell)}(g)\pi_{j1}^{(\ell)}(b)$. As $\pi_{j1}^{(\ell)}(b) \in \mathcal{M}$ for any $1 \leq j \leq n_\ell$, we conclude that $\rho(g)(p)$ is in the span of $\mathcal{M}$. Hence the span of $\mathcal{M}$ is invariant under the action of ρ. □

Proposition 5.2. *The set $\mathcal{Q}$ built it in Algorithm 1 is a symmetry adapted basis of the least interpolation space $\Lambda_\downarrow$.*

Algorithm 1 Invariant interpolation space

In: $\mathcal{P} = \bigcup_{\ell=1}^{\mathfrak{n}} \mathcal{P}^{(\ell)}$ and $\mathcal{L} = \bigcup_{\ell=1}^{\mathfrak{n}} \mathcal{L}^{(\ell)}$ symmetry adapted bases of $\mathbb{K}[\mathrm{x}]_{\leq d}$ and Λ respectively.
Out: - a symmetry adapted basis $\mathcal{M}$ of an invariant interpolation space of minimal degree
- a symmetry adapted basis $\mathcal{Q}$ of the least interpolation space $\Lambda_{\downarrow}$.

1: **for** $\ell = 1$ `to` $\mathfrak{n}$ **do**
2: $\quad \mathrm{L}^{(\ell)}\mathrm{U}^{(\ell)} := \mathrm{W}_{\mathcal{L}^{(\ell,1)}}^{\mathcal{P}^{(\ell,1)}}$; with $\mathrm{U}^{(\ell)} = \left(u_{ik}^{(\ell)}\right)_{i,k}$ ▷ LU factorization of $\mathrm{A}^{(\ell)} = \mathrm{W}_{\mathcal{L}^{(\ell,1)}}^{\mathcal{P}^{(\ell,1)}}$
3: $\quad J^{(\ell,1)} := (j_1, \ldots, j_{r_\ell})$; ▷ echelon idex sequence of $\mathrm{U}^{(\ell)}$
4: $\quad J^{(\ell)} \leftarrow \bigcup_{k=0}^{n_\ell - 1} \{j_1 + km_\ell, j_2 + km_\ell, \ldots, j_{r_\ell} + km_\ell\}$;
5: $\quad \mathcal{M}^{(\ell)} \leftarrow \left\{p_i \;:\; p_i \in \mathcal{P}^{(\ell)} \text{ and } i \in J^{(\ell)}\right\}$;
6: $\quad \mathcal{Q}^{(\ell)} \leftarrow \left\{ \sum_{\deg(p_k)=\deg(p_i)} u_{ik}^{(\ell)} \overline{p}_k^{\dagger} \;:\; p_k \in \mathcal{P}^{(\ell)} \text{ and } i \in J^{(\ell)} \right\}$;
7: $\mathcal{M} \leftarrow \bigcup_{i=1}^{\mathfrak{n}} \mathcal{M}^{(\ell)}$ and $\mathcal{Q} \leftarrow \bigcup_{i=1}^{\mathfrak{n}} \mathcal{Q}^{(\ell)}$;
8: **return** $(\mathcal{M}, \mathcal{Q})$;

Proof. By Proposition 2.3 we get that $\mathcal{Q}$ is a basis of $\Lambda_{\downarrow}$. Let $\mathcal{Q}^{(\ell,j)} = \{q_{1,j}^{(\ell)}, \ldots, q_{m_\ell,j}^{(\ell)}\} = \mathbb{K}[\mathrm{x}]_d{}^{(\ell,j)} \bigcap \mathcal{Q}$ with $1 \leq j \leq n_\ell$. By the block diagonal structure and Corollary 4.2 we have

$$q_{i,j}^{(\ell)} = \sum_k u_{ik}^{(\ell)} \left(\pi_{j1}^{(\ell)}\left(\overline{p}_k^{(\ell)}\right)\right)^{\dagger} = \pi_{j1}^{(\ell)}\left(\sum_k u_{ik}^{(\ell)} \left(\overline{p}_k^{(\ell)}\right)^{\dagger}\right) = \pi_{j1}^{(\ell)}\left(q_{i,1}^{(\ell)}\right).$$

Therefore $\mathcal{Q}^{(\ell)}$ has the following structure

$$\mathcal{Q}^{(\ell)} = \left\{q_1^{(\ell)}, \ldots, q_{r_\ell}^{(\ell)}; \ldots; \pi_{n_\ell 1}^{(\ell)}\left(q_1^{(\ell)}\right), \ldots, \pi_{n_\ell 1}^{(\ell)}\left(q_{r_\ell}^{(\ell)}\right)\right\}.$$

Since for any $1 \leq i \leq r_\ell$, $q_i^{(\ell)}, \pi_{21}^{(\ell)}(q_i^{(\ell)}), \ldots \pi_{n_\ell 1}^{(\ell)}(q_i^{(\ell)})$ form a basis of an irreducible representation of $\mathcal{G}$ we can conclude that $\mathcal{Q}$ is a symmetry adapted basis of $\Lambda_{\downarrow}$. □

If the linear representation $\varrho : \mathcal{G} \to \mathbb{R}^n$ is orthogonal, the apolar product is $\mathcal{G}-$invariant. As pointed out in Section 4.1, we can construct a symmetry adapted basis $\mathcal{P}$ of $\mathbb{K}[\mathrm{x}]_d$ that is orthonormal. Then $\mathcal{P} = \mathcal{P}^{\dagger}$ and the basis $\mathcal{M}$ built in Algorithm 1 is orthonormal. Moreover if in the third step of Algorithm 1 we use Gaussian Elimination by segment as in (De Boor and Ron, 1992a), then $\mathcal{Q}$ is an orthonormal symmetry adapted basis of $\Lambda_{\downarrow}$.

With this construction we reproved that $\Lambda_{\downarrow}$ is invariant. The above approach to computing a basis of $\Lambda_{\downarrow}$ is advantageous in two ways. First Gaussian elimination is performed only on smaller blocks. But also, when solving invariant and equivariant interpolation problems, the result will respect exactly the intended invariance or equivariance, despite possible numerical inaccuracy.

5.2. Computing interpolants

We consider an interpolation problem (Λ, ϕ) where Λ is a $\mathcal{G}$-invariant subspace of $\mathbb{K}[\mathrm{x}]^*$ and $\phi : \Lambda \to \mathbb{K}^m$. Take $\boldsymbol{Q}$ to be a symmetry adapted basis of an invariant interpolation space $\mathcal{Q}$ for Λ as obtained from Algorithm 1. The interpolant polynomial q that solves (Λ, ϕ) in $\mathcal{Q}$ is given by

$$q = \sum_{\ell=1}^{\mathrm{n}} \sum_{i=1}^{n_\ell} \boldsymbol{Q}^{(\ell,i)} \left(\mathrm{A}^{(\ell)}\right)^{-1} \phi(\mathcal{L}^{(\ell,i)})^t, \tag{5.1}$$

where $\boldsymbol{Q}^{(\ell,i)}$, $\mathcal{L}^{(\ell,i)}$ are as in (4.5) and $\mathrm{A}^{(\ell)} = \mathrm{W}_{\mathcal{L}^{(\ell,1)}}^{\boldsymbol{Q}^{(\ell,1)}}$. Note that we made no asumption on ϕ. The invariance of Λ allows us to decompose the problem into smaller blocks, independently of the structure of ϕ. This illustrates how symmetry can be used to better organize computations : if we can choose the points of evaluation, the computational cost can be alleviated by choosing them with some symmetry.

When ϕ is invariant or equivariant, Equation (5.1) can be further reduced. If (Λ, ϕ) is an invariant interpolation problem, it follows from Schur's lemma that $\phi(\mathcal{L}^{(\ell)}) = 0$ for any $\ell > 1$. Therefore for solving any invariant interpolation problem we only need to compute the first block of $\mathrm{W}_{\mathcal{L}}^{\mathcal{P}}$, i.e., the interpolant is given by $\boldsymbol{Q}^{(1)} \left(\mathrm{A}^{(1)}\right)^{-1} \phi(\mathcal{L}^{(1)})^t$.

More generally if (Λ, ϕ) is a $\varrho-\theta$ equivariant problem, such that the irreducible representation $\mathrm{r}^{(\ell)}$ does not occur in θ, then $\phi(\mathcal{L}^{(\ell)}) = 0$. The related block can thus be dismissed.

Example 5.2.1. Following on Example 3.1.1. Since we are dealing with an invariant interpolation problem, we only need to compute bases of $\Lambda^{\mathcal{G}}$ and $\mathbb{K}[\mathrm{x}]_{\leq 5}^{\mathcal{G}}$. We have

$$\mathcal{L}^{\mathcal{G}} = \left\{ \mathrm{e}_{\xi_1}, \sum_{i=2}^{6} \mathrm{e}_{\xi_i}, \sum_{i=7}^{11} \mathrm{e}_{\xi_i}, \sum_{i=12}^{16} \mathrm{e}_{\xi_i} \right\}$$

and

$$\mathcal{P}^{\mathcal{G}} = \{1,\ x_1^2 + x_2^2,\ x_1^4 + 2x_1^2x_2^2 + x_2^4,\ x_1^5 - 10x_1^3x_2^2 + 5x_1x_2^4\}.$$

Since $\mathrm{W} = \mathrm{W}_{\mathcal{L}^{\mathcal{G}}}^{\mathcal{P}^{\mathcal{G}}}$ is a square matrix with full rank, $\mathrm{span}_{\mathbb{K}}(\mathcal{P}^{\mathcal{G}})$ contains a unique invariant interpolant for any invariant interpolation problem. It has to be the least interpolant.

For ϕ given in Example 3.1.1, one finds the interpolant p by solving the 4×4 linear system $\mathrm{W}\, a = \phi(\mathcal{L}^G)$. The solution $a = (-0.3333333, 3.295689, -36.59337, 45.36692)^t$ provides the coefficients of $\mathcal{P}^{\mathcal{G}}$ in p. The graph of p is shown in Figure 1. If p given above is only an approximation of the least interpolant, due to numerical inaccuracy, it is at least exactly invariant. Had we computed the least interpolant with the algorithm of (De Boor and Ron, 1992a), *i.e.,* by elimination of the Vandermonde matrix based on the monomial basis, the least interpolant obtained would not be exactly invariant because of the propagation of numerical inacurracies.

We define the deviation from invariance (InvD) of $p = \sum_{\deg \alpha \leq 5} a_\alpha x^\alpha$ as

$$\sigma\,(a_{20}, a_{02}) + \sigma\left(a_{40}, \frac{a_{22}}{2}, a_{04}\right) + \sigma\left(a_{50}, -\frac{a_{32}}{10}, \frac{a_{14}}{5}\right) + \sum_{\beta \in \mathcal{B}} |a_\beta|$$

where σ is the standard deviation, and $\mathcal{B}$ represents the exponents of the monomials that do not belong to any of the elements in $\mathcal{P}^G$. In Table 1 we show the InvD for the interpolant p computed with different precisions. The obtained polynomials are somehow far from being $\mathcal{G}-$invariant.

# Digits	10	15	20	30
InvD	72.9614	40.0289	6.0967	$< 10^{-9}$

Table 1: InvD values for different digits of precision

In the same spirit, let us mention that the condition number of W_Λ^M, where M is the monomial basis of $\mathbb{K}[\mathrm{x}]_{\leq 5}$, is more than 10^2 times the condition number of $\mathrm{W}_{\mathcal{L}^G}^{\mathcal{P}^G}$. This is an indicator that two additional digits of precision are lost in the computation.

Example 5.2.2. Following up on Example 4.3.1. Let θ be the permutation representation of D_3 in $\mathbb{R}^3$. θ decomposes into two irreducible representations, the trivial representation and the irreducible representation ϑ of dimension 2. Let $\phi : \Lambda \to \mathbb{R}^3$ a $\vartheta - \theta$ equivariant map determined by $\phi(\mu_1) = (1, -1, 5)^t$. For solving (Λ, ϕ) we need only consider the first and third block of the Vandermonde matrix computed in Example 4.3.1. The $\rho^* - \theta$ equivariant map that solves (Λ, ϕ) is $\mathrm{p} = (p_1, p_2, p_3)$ with:

$$p_1 := \frac{705}{4256}x_1^2 + \frac{135}{4256}x_2^2 + \frac{31}{56}\sqrt{3}x_1 + \frac{93}{56}x_2 - \frac{15}{112}\sqrt{3}x_1x_2$$

$$p_2 := \frac{705}{4256}x_1^2 + \frac{135}{4256}x_2^2 + \frac{31}{56}\sqrt{3}x_1 - \frac{93}{56}x_2 + \frac{15}{112}\sqrt{3}x_1x_2$$

$$p_3 := -\frac{75}{2128}x_1^2 + \frac{495}{2128}x_2^2 - \frac{31}{28}\sqrt{3}x_1.$$

In Figure 3 we show the image of $\mathbb{R}^2$ by the polynomial map p and the tangency conditions imposed by ϕ.

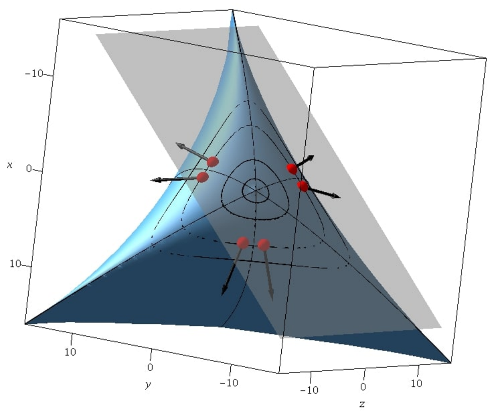


Figure 3: Parameterized surface with tangency constraints.

Example 5.2.3. This example illustrates the use of an equivariant interpolating map to parameterize a surface with an symmetric set of tangency conditions. It follows up on Example 3.2.1. Since the representation ϱ of D_3 in $\mathbb{R}^2$ is irreducible, for computing any $\varrho - \varrho$ equivariant we only need to compute the third isotopic block in the Vandermonde matrix $W_{\mathcal{L}^{(3)}}^{Q^{(3)}}$, where Q is a basis for the interpolation space Q built by Algorithm 1. This block is $W = \begin{pmatrix} A^{(3)} & \\ & A^{(3)} \end{pmatrix}$. The rows correspond to

$$\mathcal{L}^{(3)} := \left\{\mathcal{L}^{(3,1)}, \mathcal{L}^{(3,1)}\right\}, \mathcal{L}^{(3,1)} := \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\} \text{ and } \mathcal{L}^{(3,1)} := \{\lambda_5, \lambda_6, \lambda_7, \lambda_8\}$$

$$\begin{array}{ll}
\lambda_1 = \mathrm{e}_{\xi_1} + \mathrm{e}_{\xi_2} - \mathrm{e}_{\xi_4} - \mathrm{e}_{\xi_5} & \lambda_5 = \frac{\sqrt{3}}{2}(-\mathrm{e}_{\xi_1} + \mathrm{e}_{\xi_2} + 2\mathrm{e}_{\xi_3} + \mathrm{e}_{\xi_4} - \mathrm{e}_{\xi_5} - 2\mathrm{e}_{\xi_6}), \\
\lambda_2 = \mathrm{e}_{\xi_3} - \mathrm{e}_{\xi_4} - \mathrm{e}_{\xi_5} + \mathrm{e}_{\xi_6}, & \lambda_6 = \frac{\sqrt{3}}{2}(-2\mathrm{e}_{\xi_1} + 2\mathrm{e}_{\xi_2} + \mathrm{e}_{\xi_3} - \mathrm{e}_{\xi_4} - \mathrm{e}_{\xi_5} - \mathrm{e}_{\xi_6}). \\
\lambda_3 = \mathrm{e}_{\xi_7} + \mathrm{e}_{\xi_8} - \mathrm{e}_{\xi_{10}} - \mathrm{e}_{\xi_1 1} & \lambda_7 = \frac{\sqrt{3}}{2}(-\mathrm{e}_{\xi_7} + \mathrm{e}_{\xi_8} + 2\mathrm{e}_{\xi_9} + \mathrm{e}_{\xi_{10}} - \mathrm{e}_{\xi_1 1} - 2\mathrm{e}_{\xi_{12}}), \\
\lambda_4 = \mathrm{e}_{\xi_9} - \mathrm{e}_{\xi_{10}} - \mathrm{e}_{\xi_1 1} + \mathrm{e}_{\xi_{12}}, & \lambda_8 = \frac{\sqrt{3}}{2}(-2\mathrm{e}_{\xi_7} + 2\mathrm{e}_{\xi_8} + \mathrm{e}_{\xi_9} - \mathrm{e}_{\xi_{10}} - \mathrm{e}_{\xi_1 1} - \mathrm{e}_{\xi_{12}}).
\end{array}$$

The columns correspond to

$$Q^{(3)} := \left\{ \begin{array}{c} Q^{(3,1)} := \left\{x, x^2 - y^2, x^3 + xy^2, x^4 - y^4\right\}, \\ Q^{(3,2)} := \left\{y, -2xy, y(x^2 + y^2), -2xy(x^2 + y^2)\right\} \end{array} \right\}.$$

$$A^{(3)} = -\frac{2}{27} \begin{pmatrix} 72\sqrt{3} & -288 & 608\sqrt{3} & -2432 \\ 9\sqrt{3} & 90 & 76\sqrt{3} & 760 \\ 45\sqrt{3} & -90 & 140\sqrt{3} & 280 \\ 9\sqrt{3} & 18 & 28\sqrt{3} & 504 \end{pmatrix}$$

We thus determine that the equivariant interpolant for the interpolation problem described in Example 3.2.1 is given by :

$$p_1 = \frac{\alpha}{320}x + \frac{3\beta}{640}(x^2 - y^2) + \frac{9\gamma}{8960}x(x^2 + y^2) + \frac{27\delta}{17920}(x^4 - y^4)$$

$$p_2 = \frac{\alpha}{320}y - \frac{3\beta}{320}xy + \frac{9\gamma}{8960}y(x^2 + y^2) - \frac{27\delta}{8960}xy(x^2 + y^2)$$

where

$$\alpha = \sqrt{3}(25a - 114b) + 494d - 185c,\ \beta = \sqrt{3}(114d - 25c) + 38b - 5a,$$
$$\gamma = \sqrt{3}(42b - 25a) + 185c - 182d,\ \delta = \sqrt{3}(25c - 42d) + 5a - 14b.$$

Example 5.2.4. We now seek an invariant implicit surface through a given symmetric set of points. We thus seek an invariant polynomial that is zero on the given points. In order to obtain a non zero polynomial though we need to select a point, for instance the origin, where the polynomial shall not be zero.

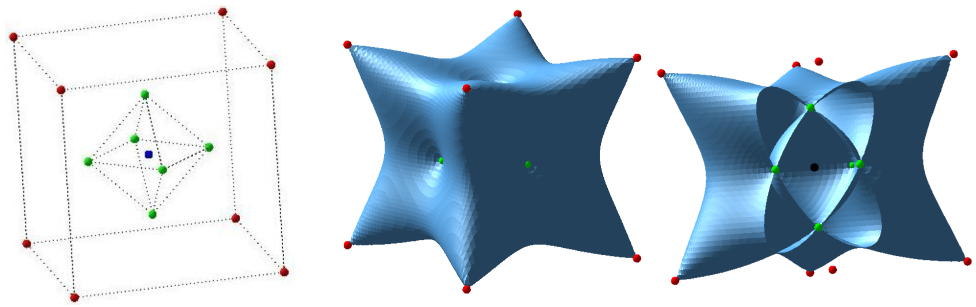
The group here is O_h, the subgroup of the orthogonal group $O_3(\mathbb{R})$ that leaves the cube invariant. It has order 48 and 10 inequivalent absolutely irreducible representations whose dimensions are $(1, 1, 1, 1, 2, 2, 3, 3, 3, 3)$.

Consider $\Xi \subset \mathbb{R}^3$ the invariant set of 15 points illustrated on Figure 4a. They are grouped in three orbits O_1, O_2 and O_3 of O_h. The orbit O_1 consists of a single point, the origin. The points in O_2 and in O_3 are the vertices and centers of the faces of cubes with the center at the origin and edge length $\sqrt{3}$ and 1 respectively.

Consider $\Lambda = \operatorname{span}\left(\left\{e_\xi \mid \xi \in \Xi\right\} \bigcup \left\{e_\xi \circ D_{\vec{\xi}} \mid \xi \in O_2 \cup O_3\right\}\right)$. Λ is an invariant subspace. Take the map $\phi : \Lambda \to \mathbb{R}$ defined by $\phi(\xi) = 3$ if $\xi \in O_1$, $\phi(\xi) = 0$ if $\xi \in O_2 \cup O_3$ and $\phi\left(e_\xi \circ D_{\vec{\xi}}\right) = 0$ if $\xi \in O_2 \cup O_3$. The pair (Λ, ϕ) is an invariant interpolation problem. The polynomial $p \in \mathbb{K}[x]^{\mathcal{G}}$ given by

$$p = 3 - \frac{20}{3}\left(x^2 + y^2 + z^2\right) + \frac{13}{3}\left(x^4 + y^4 + z^4\right) - \frac{2}{9}\left(x^2y^2 + x^2z^2 + y^2z^2\right) - \frac{2}{3}\left(x^6 + y^6 + z^6\right)$$

is the solution of (Λ, ϕ) in $\Lambda_\downarrow$. We built p from the Vandermonde matrix $W_{\mathcal{L}^{\mathcal{G}}}^{\mathcal{P}^{\mathcal{G}}_{\leq 6}} \in \mathbb{R}^{5\times 6}$ instead of the full Vandermonde matrix for bases of Λ and $\mathbb{K}[x]_{\leq 6}$ which is a 29×84 matrix. In Figure 4b we show the zero set of p. It has the symmetry of O_h and contains $O_1 \cup O_2$.



(a) Points in Ξ divided in orbits

(b) Variety of p

Figure 4: Interpolation data and variety of the interpolant p that goes through the red and green points.

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